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PARAMETRIC REPRESENTATIONS OF SURFACES CONTAINING A ISOPHOTE CURVE IN 3-DIMENSIONAL GALILEAN SPACE

Şeyda Özel[1], Mehmet Bektaş[2]
Fırat University, Faculty of Science, Deparment of Mathematics, 23100, Elazığ, Turkey
Email: s_demir2323@outlook.com[1], mbektas@firat.edu.tr[2]
Corresponding Author: Şeyda Özel email: s_demir2323@outlook.com

Abstract. This paper investigates a family of defined surfaces that share a common isotope curve in three-dimensional Galilean space. By employing a given curve in this space with its Frenet frame, we derive characterizations of the surfaces and present illustrative examples where in the curve functions, as an isotope.

Keywords: The isotope curve, The non-isotope curve, The isotope-asymptotic curve, Galilean 3-space.

A. Introduction

One practical method for analyzing and visualizing surfaces based on equal-intensity lines is to use the concept of isophote curves. Isophote curves can be considered a subset of surface-defining curves, like geodesics or curvature lines, characterized by the constant-angle relation between their normals and a chosen fixed direction. In addition to its appeal, the isophote curve is widely employed in computer-graphics applications and remains a local point of contemporary research in geometry.

An isphote curve on a given surface is determined in two steps: First, compute the surface's normal vector field $n(s,t)$; then, trace the points on the surface that satisfy

$$\frac{\langle n(s,t), d\rangle}{\|n(s,t)\|} = cos\beta,$$

where β denotes an angle constrained to the interval $0 < \beta < \frac{\pi}{2}$. When the quantity

$$\frac{\langle n(s,t), d\rangle}{\|n(s,t)\|} = 0$$

evaluates to zero, given that d it is the unit fixed vector, the isophote curve is referred to as a silhouette curve (Doğan & Yaylı, 2015). Research on curves and their properties dates back to the earliest studies. Koenderink and van Doorn (1980) investigated image brightness contours, which correspond to isophote curves. Poeschl (1984) utilized isophote curves to identify geometric inconsistencies on free-form body designs. Sara (1994) investigated how a surface's local shading behavior can be inferred from the characteristics of its curves. Grounded in the theory of surfaces, his work zeroed in on precisely gauging the tilt of the surface normal while also tackling the qualitative reconstruction of Gaussian curvature. In a study, Kim and Lee (2003) tackled the parameterization of isophote curves on both rotational and canal surfaces. By capitalizing on the insight that each of these shapes can be decomposed into a family of circles, they demonstrated that the normals, at points lying on any given circle. Takeuchi (2004) introduced the concept of a helix, defining it as a space curve whose principal normal lines consistently maintain a fixed angle with a chosen direction. In a study, Dogan (2012) examined isophote curves on timelike surfaces, within the Minkowski space E_1^3. The problem of surface families sharing a line of curvature has been addressed in (Ergün, Bayram and Kasap, 2014; Ergün, Bayram and Kasap, 2015). Galilean geometry, derived from the Galilean principle of

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relativity, forms a distinct branch of classical geometry (Yaglom, 2012). It turns out that Galilean geometry enjoys a wide array of practical applications in the physical sciences (Musielak & Fry, 2019). Curves and surfaces in Galilean geometry have been dealt with by many researchers (Aydın, Külahçı & Öğrenmiş, 2019; Dede, 2013; Diviak & Sipus et al., 2002). In surface theory and physics, geodesics play a critical role. A curve on the surface is called a geodesic if its geodesic curvature is identically zero. To put it another way, a curve's normal vector is always parallel to the surface's normal vector. Further research and observations on surfaces in $\mathbb{G}_3$ have been reported. By definition, a line of curvature is a curve that lies on a surface and whose tangent direction aligns with the principal curvature direction at every point. An asymptotic condition appears when the binormal $B(s)$ of $\alpha(s)$ and the regular $n(s, s_0)$ of the surface become parallel at any location on $\alpha(s)$. Moreover, additional studies on families of surfaces sharing asymptotic curves can be found in (Yoon, Yüzbaşı & Bektaş, 2017; Yüzbaşı, 2016). Recent studies have shown growing interest in isophote curves and their geometric properties in non-Euclidean and Galilean spaces (Ali and Turgut, 2019; Celik and Onder, 2020; Ersoy and Tosun, 2021; Yoon, 2022). In addition, Körpınar and Demir (2018) investigated isophote curves on surfaces in non-Euclidean settings. Motivated by these works, we study isophote curves on admissible regular surfaces in Galilean 3-space. Despite the extensive literature on isophote curves in Euclidean and Minkowski geometries, similar investigations in Galilean 3-space are scarce. To the best of our knowledge, the problem of characterizing surfaces containing isophote curves, particularly timelike surfaces, has not been systematically addressed in the Galilean setting. This lack of results constitutes the main motivation of the present study. Accordingly, we aim to fill this gap by analyzing isophote curves in Galilean 3-space and providing explicit parametric representations of surfaces that admit such curves. Although isophote curves have been extensively studied in Euclidean and Minkowski spaces, their characterization in Galilean 3-space remains largely unexplored. In particular, the behavior of isophote curves on timelike surfaces within the Galilean framework has not been adequately addressed. Motivated by this gap, the present study investigates parametric representations of surfaces containing an isophote curve in 3-dimensional Galilean space. By deploying the Frenet frame adapted to Galilean geometry, necessary and sufficient criteria are derived that determine when a given curve qualifies as an isophote, a non-isophote, or a silhouette curve.

B. Notations

$\mathbb{G}_3$ designates a Galilean space, the three-dimensional complex projective P_3, endowed with a distinguished real plane, w of ideal planes a real line $f \subset w$ that collects the ideal lines, and an ideal form w, f, I_1, I_2 from which the ideal elements $I_1$and I_2 emerge, both of which line of f. Vectors in the Galilean space are classified as either isotropic or non-isotropic.

Definition 1. When the first component of a vector $x = (x_1, x_2, x_3)$ is non-zero $(x_1 \neq 0)$, the vector is called non-isotropic. Each unit isotropic vector can be expressed as $x = (1, x_2, x_3)$. If the leading entry vanishes $x_1 = 0$, the vector is isotropic; otherwise, it remains non-isotropic.

Definition 2. Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be vectors in $\mathbb{G}_3$, Galilean space. The Galilean scalar product of the two vectors is defined as

$$\langle x, y\rangle = \{x_1 y_1 | x_1 \neq 0\}$$
$$\langle x, y\rangle = \{x_2 y_2 + x_3 y_3 | x_1 = 0\}.$$

Definition 3. Let $x = (x_1, x_2, x_3)$ and $y = (y, y_2, y_3)$ be vectors, in the Galilean space $\mathbb{G}_3$. The Galilean vector product between two vectors is given by

$$x \wedge y = \begin{vmatrix} 0 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (0, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

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Here e_2 is the vector (0,1,0) while e_3 is the vector (0,0,1).
Consider a curve $\alpha(s)$ in dimensional Galilean space with the three-($\alpha > 3$) parametrised by the variable s. In coordinate form the curve is written as $\alpha(s) = (s, f(s), g(s))$. Its curvature $\kappa(s)$ is obtained by taking the norm of the second-order derivatives of the transverse components:

$$\kappa(s) = \sqrt{f''(s)^2 + g''(s)^2}.$$

The torsion $\tau(s)$ is given by the triple product of the first three derivatives divided by the square of the curvature:

$$\tau(s) = \frac{\det(\alpha'(s), \alpha''(s), \alpha'''(s)}{\kappa^2(s)}.$$

Moreover, the curve fulfills its moving trihedron.
The tangent vector at the parameter s is $T(s) = \alpha'(s) = (1, f'(s), g'(s))$. The principal normal follows as $N(s) = \frac{\alpha''(s)}{\kappa(s)} = \frac{1}{\kappa(s)}(0, f''(s), g''(s))$. The binormal orthogonal, to both T and N is $B(s) = \frac{1}{\kappa(s)}(0, -g''(s), f''(s))$. The curve $\alpha(s)$ carries the tangent, binormal vectors, denoted by the ordered triple $\{T, N, B\}$. In these terms, the Frenet–Serret formulas are $T' = \kappa. N, N' = \tau. B, B' = -\tau. N$.

Definition 4. Assume that $P(s,t)$ is a surface, in the Galilean space $\mathbb{G}_3$. For parameters $s, t \in \mathbb{R}$, the surface can be expressed by

$$P(s,t) = \alpha(s) + \lambda_1(s,t)P_1(s,t) + \lambda_2(s,t)P_2(s,t) + \lambda_3(s,t)P_3(s,t).$$

In this case, each of the component functions $P_1(s,t), P_2(s,t), P_3(s,t)$ belongs to the class C^3. Moreover, the surface carries a normal vector field, which we write as $n(s,t)$; here P_s and P_t denote the partial derivatives of P with respect to s and t, respectively.

Definition 5. A curve α(s) lying on S is classified as follows:

i) **Curvature line**: Its tangent vector aligns with a principal curvature direction.
ii) **Geodesic:** The curve's normal vector field $N(s)$ is parallel to the surface normal at all points.
iii) **Asymptotic curve:** The curve $\alpha(s)$ is called asymptotic if at every point on it, the binormal $B(s)$ aligns with the surface's $n(s, s_0)$ making the two vectors parallel.

When $\alpha(s)$ functions as a curve that is also parametric, it is termed isoasymtotic; and when $\alpha(s)$ serves as a geodesic curve with a parametric description, the designation is isogeodesic.

C. Parametric Represantations of Surfaces Containing a Common Isophote curve In 3-Dimensional Galilean space

In what follows we present a series of characterizations and illustrative examples of surfaces that admit this curve as an isophote drawing on a curve situated in three-dimensional Galilean space with its Frenet frame.

Case 1. Let d be a fixed unit isotropic vector. In order for a isoparametric α curve to serve as an isophote curve on the surface of $P(s,t)$, we must extract the spesific conditions that the surface must satisfy. First observe that the curve $\alpha(s)$ is a curve on $P(s,t)$ for the parameter value $t_0 \in [0, J]$. Across the interval $0 \le s \le 1$ the three functions

$$\lambda_1(s,t_0) = \lambda_2(s,t_0) = \lambda_3(s,t_0) = 0. \quad (1)$$

Equality is attained. Consequently the computation utilizes the equality that meets the necessary-and-sufficient condition

$$\langle n(s,t), d\rangle = cos\theta \quad (2)$$

as well as the surface normal equation (3) which defines $n(s,t)$ as the product of the derivative of $P(s,t)$ with respect to s and the partial derivative and $P(s,t)$ with respect to t.

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$$n(s,t) = \frac{\partial P(s,t)}{\partial s} \times \frac{\partial P(s,t)}{\partial t}. \quad (3)$$

For the curve $\alpha(s)$ to line on the surface $P(s,t)$, the partial derivatives $P(s,t)$ with respect to s and t are computed using the Frenet frame (T,N,B) of the curve. Differentiating $P(s,t)$ with respect to s yields

$$\frac{\partial P(s,t)}{\partial s} = \left[1 + \frac{\partial \lambda_1(s,t)}{\partial s}\right] T(s) + \left[\kappa.\lambda_1(s,t) + \frac{\partial \lambda_2(s,t)}{\partial s} - \tau.\lambda_3(s,t)\right] N(s,t) + \left[\lambda_2(s,t).\tau + \frac{\partial \lambda_3(s,t)}{\partial s}\right] B(s) \quad (4)$$

while differentiation with respect to t gives

$$\frac{\partial P(s,t)}{\partial t} = \frac{\partial \lambda_1(s,t)}{\partial t}.T(s) + \frac{\partial \lambda_2(s,t)}{\partial t}.N(s) + \frac{\partial \lambda_3(s,t)}{\partial t}.B(s). \quad (5)$$

Substituting equation (4) and (5) into equation (3) and using the properties of the Frenet frame, the surface normal vector can be expressed as a linear combination of the normal and binormal vectors, leading to Equation (6)

$$n(s,t) = \left[-\left(1 + \frac{\partial \lambda_1(s,t)}{\partial s}\right)\frac{\partial \lambda_3(s,t)}{\partial t} + \left(1 + \frac{\partial \lambda_1(s,t)}{\partial s}\right)\frac{\partial \lambda_3(s,t)}{\partial t}\right] N(s) + \left[\left(1 + \frac{\partial \lambda_1(s,t)}{\partial s}\right)\frac{\partial \lambda_2(s,t)}{\partial t} - \left(\frac{\partial \lambda_2(s,t)}{\partial s}\right)\frac{\partial \lambda_1(s,t)}{\partial t}\right] B(s). \quad (6)$$

Since $\alpha(s)$ is an isoparametric curve on the surface, we consider the restriction $t = t_0$.Consequently, the surface normal along the curve can be written in the form

$$n(s,t_0) = \phi_1(s,t_0).T(s) + \phi_2(s,t_0).N(s) + \phi_3(s,t_0).B(s). \quad (7)$$

In this case the coefficient functions line up like this:

$$\phi_1(s,t_0) = 0 \quad (8)$$
$$\phi_2(s,t_0) = \left[-\left(1 + \frac{\partial \lambda_1(s,t_0)}{\partial s}\right)\frac{\partial \lambda_3(s,t_0)}{\partial t} + \left(1 + \frac{\partial \lambda_1(s,t_0)}{\partial s}\right)\frac{\partial \lambda_3(s,t_0)}{\partial t}\right]$$
$$\phi_3(s,t_0) = \left[\left(1 + \frac{\partial \lambda_1(s,t_0)}{\partial s}\right)\frac{\partial \lambda_2(s,t_0)}{\partial t} - \left(\frac{\partial \lambda_2(s,t_0)}{\partial s}\right)\frac{\partial \lambda_1(s,t_0)}{\partial t}\right].$$

Furthermore because the curve is an isoparametric curve on the surface, $\partial\lambda_1(s,t_0) = \partial\lambda_2(s,t_0) = \partial\lambda_3(s,t_0) = 0$ the partial derivatives of the functions $\lambda_1, \lambda_2, \lambda_3$ according to parameter s are equal to zero. Consequently the preceding expression can be rearranged as follows:

$$\phi_1(s,t_0) = 0 \quad (9)$$
$$\phi_2(s,t_0) = \frac{\partial \lambda_3(s,t_0)}{\partial t}$$
$$\phi_3(s,t_0) = \frac{\partial \lambda_2(s,t_0)}{\partial t}.$$

Substituting these expressions into equation (7), the surface normal along the curve is obtained as

$$n(s,t_0) = -\frac{\lambda_3(s,t_0)}{\partial t} N(s) + \frac{\partial \lambda_2(s,t_0)}{\partial t} B(s). \quad (10)$$

With the above operations, we have obtained the surface normal along the curve.

$$d = \left(-\frac{k_n}{k}\cos\theta - \frac{k_g}{k}\sin\theta\right)N(s) + \left(-\frac{k_n}{k}\sin\theta + \frac{k_g}{k}\cos\theta\right)B(s). \quad (11)$$

Therefore, using the isophote condition, we can write as following:

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$$\frac{\langle n(s,t_0),d\rangle}{\|n(s,t_0)\|} = \frac{1}{\sqrt{\phi_2^2(s,t_0)+\phi_3^2(s,t_0)}}\langle \frac{\partial\lambda_3(s,t_0)}{\partial t}N(s) + \frac{\partial\lambda_2(s,t_0)}{\partial t}B(s), (-\frac{k_n}{k}cos\theta - \frac{k_g}{k}\sin\theta)N(s) + \left(-\frac{k_n}{k}sin\theta + \frac{k_g}{k}cos\theta\right)B(s)\ \rangle. \quad (12)$$

When all of the requisite steps are taken, the result is obtained as following;

$$\frac{\langle\ n(s,t_0),d\rangle}{\|\ n(s,t_0)\|} = \frac{1}{\sqrt{\phi_2^2(s,t_0)+\phi_3^2(s,t_0)}}[sin\phi cos\theta\phi_2(s,t_0) + cos\phi cos\theta\phi_3(s,t_0)] \quad (13)$$

Considering equations (1) and (2) we can obtain the following theorem with a simple calculation.

Theorem 1. Let $\alpha(s)$ be a curve on an admissible regular surface $P(s,t)$. The curve is both an isoparametric and an isophote curve if and only if the following relations are satisfied:

$$\lambda_1(s,t_0) = \lambda_2(s,t_0) = \lambda_3(s,t_0) = 0 \quad (14)$$

$$\frac{\partial\lambda_3(s,t_0)}{\partial t}cos\phi = \frac{\partial\lambda_2(s,t_0)}{\partial t} - sin\phi$$

Proof. Assume that the curve $\alpha(s)$ is both an isoparametric and an isophote curve on the admissible regular surface $P(s,t)$. Since $\alpha(s)$ is isoparametric, it lies on a parameter curve $t = t_0$, which implies

$$\lambda_1(s,t_0) = \lambda_2(s,t_0) = \lambda_3(s,t_0) = 0\,. \quad (15)$$

Moreover, the isophote condition requires that the surface normal makes a constant angle ϕ with a fixed direction. Differentiating this condition with respect to t and evaluating at $t = t_0$, we obtain

$$\frac{\partial\lambda_3(s,t_0)}{\partial t} = -sin\phi$$

$$\frac{\partial\lambda_2(s,t_0)}{\partial t} = cos\phi,$$

which leads to relations given in (14) and (15).

Conversely, suppose that the conditions

$$\lambda_1(s,t_0) = \lambda_2(s,t_0) = \lambda_3(s,t_0) = 0$$

together with

$$\frac{\partial\lambda_3(s,t_0)}{\partial t} = -sin\phi$$

$$\frac{\partial\lambda_2(s,t_0)}{\partial t} = cos\phi,$$

are satisfied. The first condition ensures that $\alpha(s)$ is an isoparametric curve on $P(s,t)$, while the latter relations guarantee that the surface normal along $\alpha(s)$ makes a constant angle ϕ with the fixed direction. Hence, $\alpha(s)$ is an isophote curve. Therefore, the curve $\alpha(s)$ is both isoparametric and isophote if and only if the stated conditions hold.

Theorem 2. Suppose the curve $\alpha(s)$ is isoparametric on the surface $P(s,t)$. It is an isophote-geodesic curve if and only if the following equations are satisfied:

$$\lambda_1(s,t_0) = \lambda_2(s,t_0) = \lambda_3(s,t_0) = 0$$

$$\frac{\partial\lambda_2(s,t)}{\partial t} = 0$$

$$\frac{\partial\lambda_3(s,t)}{\partial t} = -1; \phi = \frac{3\pi}{2}$$

Proof. The normal of surface $P(s,t)$ is equation (10). Furthermore, since the $\alpha(s)$ curve is geodesic $k_g = 0$,and if $k_g = 0$ is used in the following equation

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$$d=\left(\frac{-k_n}{k}cos\theta-\frac{k_g}{k}sin\theta\right)N(s)+\left(\frac{-k_n}{k}sin\theta+\frac{k_g}{k}cos\theta\right)B(s)$$
$$d=\frac{-k_n}{k}cos\theta.N(s)-\frac{k_n}{k}sin\theta B(s)$$
is obtained. Since $\alpha(s)$ curve is geodesic, it can be taken
$$\frac{\partial\lambda_3(s,t)}{\partial t}=-1$$
$$\frac{\partial\lambda_2(s,t)}{\partial t}=0$$

as per definition (5). Furthermore, $\phi=\frac{3\pi}{2}$ is obtained from the condition of being an isophote curve. As a result, this completes the proof.

Theorem 3. Let $\alpha(s)$ be an isoparametric curve on the surface $P(s,t)$. It is isophote-asymtotic if and only if the binormal aligns with the surface, satisfying the following equations:
$$\lambda_1(s,t_0)=\lambda_2(s,t_0)=\lambda_3(s,t_0)=0$$
$$\frac{\partial\lambda_3(s,t)}{\partial t}=0$$
$$\frac{\partial\lambda_2(s,t)}{\partial t}=1;\phi=[0,2\pi].$$

Proof. The normal of surface $P(s,t)$, which is equation (10), is as follows;
$$n(s,t_0)=-\frac{\lambda_3(s,t_0)}{\partial t}N(s)+\frac{\partial\lambda_2(s,t_0)}{\partial t}B(s)\,.$$
Here, since the $\alpha(s)$ curve is asymtotic, $k_n=0$
$$-\frac{k_g}{k}sin\theta N(s)+\frac{k_g}{k}cos\theta B(s)$$
is obtained if $k_n=0$ in the equation
$$d=\left(-\frac{k_n}{k}cos\theta-\frac{k_g}{k}sin\theta\right)N(s)+\left(-\frac{k_n}{k}sin\theta+\frac{k_g}{k}cos\theta\right)B(s).$$
Since $\alpha(s)$ curve is an asymtotic curve,
$$\frac{\partial\lambda_3(s,t)}{\partial t}=0$$
$$\frac{\partial\lambda_2(s,t)}{\partial t}=1$$
can be taken as per definition (5). Furthermore, $\phi=0,\phi=2\pi$ is obtained when the required operations are performed according to description isophote curve from the condition of being isophote curve. As a result, this completes the proof.

Theorem 4. Assume that the $\alpha(s)$ curve on the surface P(s,t) is an isoparametric curve. If and only if the curve $\alpha(s)$ to be a silhouette curve on the surface P(s,t), equations of
$$\lambda_1(s,t_0)=\lambda_2(s,t_0)=\lambda_3(s,t_0)=0$$
$$\frac{\partial\lambda_3(s,t)}{\partial t}=sin\theta$$
$$\frac{\partial\lambda_2(s,t)}{\partial t}=cos\phi;\phi=\frac{\pi}{2}$$
should be provided.

Proof. The normal of surface P(s,t), with the help of equation (10), is as follows:
$$n(s,t_0)=-\frac{\lambda_3(s,t_0)}{\partial t}N(s)+\frac{\partial\lambda_2(s,t_0)}{\partial t}B(s)\,.$$
When $d=Q$ is taken here; in case of silhouette curve, $\langle n,d\rangle=0$ can be written. If $\langle n,d\rangle=0$, in the condition of being a silhouette,

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$$\frac{\partial\lambda_3(s,t)}{\partial t} = sin\phi$$
$$\frac{\partial\lambda_2(s,t)}{\partial t} = cos\phi$$

is obtained. In the case of $d = -Q$

$$\frac{\partial\lambda_3(s,t)}{\partial t} = -sin\phi$$
$$\frac{\partial\lambda_2(s,t)}{\partial t} = -cos\phi$$

is obtained in the same way. As a result, proof is given.

Theorem 5. Assume that the $\alpha(s)$ curve on the surface $P(s,t)$ is an isoparametric curve.If and only if the $\alpha(s)$ curve to be a silhouette curve on the surface $P(s,t)$;

$$\frac{\partial\lambda_3(s,t)}{\partial t} = sin\phi$$
$$\frac{\partial\lambda_2(s,t)}{\partial t} = cos\phi;\ \phi = \frac{\pi}{2}, \frac{3\pi}{2}$$

should be provided.

Proof. The normal of surface P(s,t), with help of equation (3.10), is as follows:

$$n(s,t) = -\frac{\lambda_3(s,t_0)}{\partial t}N(s) + \frac{\partial\lambda_2(s,t_0)}{\partial t}B(s)\,.$$

Also, $d = T + Q$ and $Q = cos\phi N(s) + sin\phi B(s)$ can be written. According to the definition silhouette curve,

$$\frac{\partial\lambda_3(s,t)}{\partial t} = sin\phi$$
$$\frac{\partial\lambda_2(s,t)}{\partial t} = cos\phi$$

can be written as $\langle n, d\rangle = 0$ from equation silhouette curve definition. Also, since the $\alpha(s)$ curve is a geodesic curve,

$$\frac{\partial\lambda_2(s,t)}{\partial t} = 0$$
$$\frac{\partial\lambda_3(s,t)}{\partial t} \neq 0$$

according to definition (5). Since $\frac{\partial\lambda_2(s,t)}{\partial t} = cos\phi = 0; \phi = \frac{\pi}{2}, \frac{3\pi}{2}$ can be taken from here.

Example 1. Let $\alpha(s)$ be a parametrized by $\alpha(s) = (s, 2s^2, -2s^2)$. It is a calculate that

$$T(s) = (1, 4s, -4s)$$
$$N(s) = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$
$$B(s) = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

,where $\kappa = 4\sqrt{2}$ is the curvature and $\tau = 0$ is the torsiyon of the curve in G^3. Then we obtained the surfaces family with the common isoasymtotic. If we take

$$\frac{\partial\lambda_1(s,t)}{\partial t} = 0$$
$$\frac{\partial\lambda_2(s,t)}{\partial t} = sint + 1$$
$$\frac{\partial\lambda_3(s,t)}{\partial t} = cos^2 t$$

and $t = t_0$ such that theorem (3.3) is satisfied. So a member of this family is obtained by

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$$P(s,t)=\left(s,2s^2+1+\frac{sint+cos^2t}{\sqrt{2}},-2s^2+\frac{(-1-sint+cos^2t)}{\sqrt{2}}\right).$$

Case 2. Considering d is a unit fixed non- isotropic vector:
Theorem 6. The necesssary and sufficient condition for the $\alpha(s)$ curve to be an isoparametric and non-isotope curve on the surface of $P(s,t)$ is to provide the equations:

$$\lambda_1(s,t_0)=\lambda_2(s,t_0)=\lambda_3(s,t_0)=0$$
$$\frac{\partial\lambda_3(s,t)}{\partial t}=sin\phi$$
$$\frac{\partial\lambda_2(s,t)}{\partial t}=cos\phi$$
$$\frac{\partial\lambda_2(s,t_0)}{\partial t}.sin\phi=\frac{\partial\lambda_3(s,t)}{\partial t}=cos\phi.$$

Proof. In the situation of being non-isophote curve, $d=T+\phi.n$ and $n=-sin\phi.N+cos\phi.B$.
Due to the definition of a non-isophote curve, the surface's normal is

$$n(s,t)=\frac{-\partial\lambda_3(s,t)}{\partial t}N(s)+\frac{\partial\lambda_2(s,t)}{\partial t}B(s).$$

We can express $\lambda_1,\lambda_2,\lambda_3$ as $\langle n,d\rangle=\phi$ as follows:

$$\frac{\partial\lambda_3(s,t)}{\partial t}=sin\phi$$
$$\frac{\partial\lambda_2(s,t)}{\partial t}=cos\phi.$$

When we proportion $\frac{\partial\lambda_3(s,t)}{\partial t}=sin\phi$ and $\frac{\partial\lambda_2(s,t)}{\partial t}=cos\phi$ equations, we also get the relationship

$$\frac{\partial\lambda_2(s,t_0)}{\partial t}.sin\phi=\frac{\partial\lambda_3(s,t)}{\partial t}.cos\phi.$$

Conclusion 1. In order for the $\alpha(s)$ isoparametric curve to be a non-isotope curve on the surface of $P(s,t)$, the necessary and sufficient condition is to provide the equations:

$$\lambda_1(s,t_0)=\lambda_2(s,t_0)=\lambda_3(s,t_0)=0$$
$$\frac{\partial\lambda_3(s,t)}{\partial t}=sin\phi$$
$$\frac{\partial\lambda_2(s,t)}{\partial t}=cos\phi$$
$$\frac{\partial\lambda_2(s,t_0)}{\partial t}.sin\phi=\frac{\partial\lambda_3(s,t)}{\partial t}.cos\phi.$$

Let's look at the functions $\lambda_1,\lambda_2,\lambda_3$ and see how they can be broken into two parts for better analysis and practical applications:

$$\lambda_1(s,t)=l(s).\Lambda_1(t)$$
$$\lambda_2(s,t)=m(s).\Lambda_2(t)$$
$$\lambda_3(s,t)=n(s).\Lambda_3(t).$$

The functions $l(s),m(s),n(s),\Lambda_1,\Lambda_2,\Lambda_3$ are all C^1 functionsin here.We get the following outcome under these circumstances.
Conclusion 2. For an $\alpha(s)$ curve to be isoparametric non-isophote curves on the surface of $P(s,t)$, the necessarry and sufficient condition is :

$$\lambda_1(s,t_0)=\lambda_2(s,t_0)=\lambda_3(s,t_0)=0$$
$$\frac{\partial\lambda_3(s,t)}{\partial t}=sin\phi$$
$$\frac{\partial\lambda_2(s,t)}{\partial t}=cos\phi$$

$$\Lambda_3'.n(s) = sin\phi$$
$$\Lambda_2'.m(s) = \cos\phi.$$

Theorem 7. For an $\alpha(s)$ curve to be non-isotope geodesic on the surface of $P(s,t)$, the necessary and sufficient condition is

$$\lambda_1(s,t_0) = \lambda_2(s,t_0) = \lambda_3(s,t_0) = 0$$
$$\frac{\partial\lambda_3(s,t)}{\partial t} = \frac{1}{sin\phi};\ \phi = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}.$$

Proof. In the situation of being non-isotope, $T = d + \phi.n$ and $n = -sin\phi.N + cos\phi.B$. The surface's normal is

$$n(s,t) = \frac{-\lambda_3(s,t)}{\partial t}N(s) + \frac{\partial\lambda_2(s,t)}{\partial t}B(s).$$

We can write $\lambda_1, \lambda_2, \lambda_3$ such that $\langle n, d\rangle = \phi$ using the notion of a non-isophote curve as follows;

$$\frac{\partial\lambda_3(s,t)}{\partial t} = \frac{1}{sin\phi}.$$

Also, since the curve is geodesic, $\frac{\partial\lambda_3(s,t)}{\partial t} \neq 0, \frac{\partial\lambda_2(s,t)}{\partial t} = 0$ can be represented using the geodesic curve definition. Since $\frac{\partial\lambda_3(s,t)}{\partial t} \neq 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ can be written. As a result, the proof is completed.

Theorem 8. The necessary and sufficient condition for the $\alpha(s)$ curve to be a non-isotope asymtotic on the surface of $P(s,t)$ is to provide the equations

$$\lambda_1(s,t_0) = \lambda_2(s,t_0) = \lambda_3(s,t_0) = 0$$
$$\frac{\partial\lambda_2(s,t)}{\partial t} = \frac{1}{cos\phi};\phi = 0, \frac{\pi}{6}, 2\pi, \frac{\pi}{3}.$$

Theorem 9. The necessary and sufficient condition for the $\alpha(s)$ curve to be a non-isotope sihouette curve on the surface of $P(s,t)$ is to provide the equations

$$\lambda_1(s,t_0) = \lambda_2(s,t_0) = \lambda_3(s,t_0) = 0$$
$$\frac{\partial\lambda_3(s,t)}{\partial t} = cos\phi$$
$$\frac{\partial\lambda_2(s,t)}{\partial t} = -sin\phi.$$

Proof. In the situation of being non-isotope, $d = T + \phi.n$ and $n = -sin\phi.N + cos\phi.B$.The surface's normal is $n(s,t) = \frac{-\lambda_3(s,t)}{\partial t}N(s) + \frac{\partial\lambda_2(s,t)}{\partial t}B(s)$ such that $\langle n, d\rangle = 0$ are as follows due to the silhouette curve definition :

$$\frac{\partial\lambda_3(s,t)}{\partial t} = cos\phi$$
$$\frac{\partial\lambda_2(s,t)}{\partial t} = -sin\phi.$$

Theorem 10. The necessary and sufficient condition for the $\alpha(s)$ curve to be a non-isotope silhouette geodesic curve on the surface of $P(s,t)$ is

$$\lambda_1(s,t_0) = \lambda_2(s,t_0) = \lambda_3(s,t_0) = 0$$
$$\frac{\partial\lambda_3(s,t)}{\partial t} = 1$$
$$\frac{\partial\lambda_2(s,t)}{\partial t} = 0; \phi = 0{,}2\pi.$$

Proof. In the situation of being non-isotope, $d = T + \phi.N$ and $n = -sin\phi.N + cos\phi.B$. The surface's normal is $n(s,t) = \frac{-\lambda_3(s,t)}{\partial t}N(s) + \frac{\partial\lambda_2(s,t)}{\partial t}B(s)$. $\lambda_1, \lambda_2, \lambda_3$ such that $\langle n, d\rangle = 0$ follows can be written as follows due to being silhouette definition; in other words $\langle n, d\rangle = 0$ and $\frac{\partial\lambda_3(s,t)}{\partial t}.\phi.sin\phi = 0$. Howeever, because the curve is a geodesic curve $\frac{\partial\lambda_3(s,t)}{\partial t} \neq 0$,

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$\frac{\partial\lambda_2(s,t)}{\partial t} = 0$ can be written using the geodesic curve definition. As a result, the proof is completed.

Theorem 11. The necessary and sufficient condition for the $\alpha(s)$ curve to be a non-isotope silhouette asymptotic curve on the surface of P(s,t) is

$$\lambda_1(s,t_0) = \lambda_2(s,t_0) = \lambda_3(s,t_0) = 0$$
$$\frac{\partial\lambda_2(s,t)}{\partial t} = -1$$
$$\frac{\partial\lambda_3(s,t)}{\partial t} = 0; \phi = \frac{\pi}{2}, \frac{-\pi}{2}, \frac{3\pi}{2}, \frac{-3\pi}{2}.$$

Proof. In the situation of being non-isotope, $d = T + \phi.N$ and $n = -sin\phi.N + cos\phi.B$The surface"s normal is $n(s,t) = \frac{-\lambda_3(s,t)}{\partial t}N(s) + \frac{\partial\lambda_2(s,t)}{\partial t}B(s)$. $\lambda_1, \lambda_2, \lambda_3$ such that $\langle n, d\rangle = 0$ follows can be written as follows due to being silhouette definition; in other words $\langle n, d\rangle = 0$ and $\frac{\partial\lambda_3(s,t)}{\partial t}.\phi.cos\phi = 0$. Howeever, because the curve is a a asymtotic curve $\frac{\partial\lambda_3(s,t)}{\partial t} = 0$, $\frac{\partial\lambda_2(s,t)}{\partial t} \neq 0$ can be written using the asymtotic curve definition. As a result, we can use $\frac{\partial\lambda_2(s,t)}{\partial t} = -1$. Since $cos\phi = 0$, $\phi = \frac{\pi}{2}, \frac{-\pi}{2}, \frac{3\pi}{2}, \frac{-3\pi}{2}$ can also be used.

D. Conclusion

In the setting of three-dimensional Galilean space, the three families of curves, isophote, non-isophote, and silhouette that lie on defined surfaces are examined. By deploying the Frenet frame adapted to Galilean geometry, necessary and sufficient criteria are derived that determine when a given curve qualifies as an isophote, a non-isophote, or a silhouette curve. These results highlight the distinctive geometric behavior of isophote, non-isophote, and silhouette curves in the three-dimensional Galilean space, which differs essentially from the classical Euclidean and Minkowski settings. The criteria obtained here provide a clearer characterization of such curves within Galilean geometry. Future work may focus on extending these results to higher-dimensional Galilean spaces or to other types of curves and surfaces.

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