

AN ANALYTICAL SOLUTION OF THE GENERALIZED HEAT EQUATION USING THE OFFSET FRACTIONAL FOURIER TRANSFORM

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Abstract. In this work, the main objective is to find a solution to the general heat equation using the offset fractional Fourier transform (OFrFT), which is represents a generalized form of the solution of the heat equation using the classical Fourier transform. This method provides a broader analytical framework for solving partial differential equations such as the heat equation. Several illustrative examples are presented to demonstrate the effectiveness and power of the OFrFT in comparison with the classical Fourier transform.

Keywords: Offset fractional Fourier transform, Generalized heat equation.

A. Intoduction

The study of integral transforms has long been a cornerstone in the development of analytical and computational methods in mathematics, physics, and engineering. Among these, the classical Fourier transform has played a central role due to its powerful ability to decompose signals into frequency components, providing profound insights into the behavior of linear systems, solving differential equations, and enabling signal analysis. However, as the complexity of scientific problems has increased, so too has the need for more generalized tools. One such generalization is the fractional Fourier transform (FrFT), a powerful mathematical tool that interpolates between the time and frequency domains, offering a richer framework for analyzing and processing signals.

The offset fractional Fourier transform (OFrFT) is a further extension of the FrFT, incorporating additional degrees of freedom that allow for more nuanced transformations. These additional parameters in the OFrFT offer enhanced flexibility, enabling improved modeling capabilities in systems where conventional transformations may fall short. The OFrFT can be viewed as a shifted and rotated version of the FrFT, capable of accommodating offset phenomena that naturally arise in various physical contexts.

In recent years, the FrFT and its variants have seen rapid development and widespread adoption across several disciplines, including optics, quantum mechanics, signal processing, and applied mathematics. The fractional nature of the FrFT allows it to model intermediate domains between time and frequency, making it especially valuable in systems where chirp-like signals or nonstationary processes dominate. For instance, in optical signal processing, the FrFT has been employed to model the propagation of light through quadratic graded-index media and lens systems (Bernardo & Soares, 1994; Mendlovich & Ozaktas, 1993; Ozaktas & Aytür, 1995; Ozaktas & Kutay, 2001; Liu et al., 1997). These works have demonstrated the



superiority of the FrFT in describing diffraction patterns, lens focusing behavior, and imaging systems under conditions where the classical Fourier framework proves inadequate.

In the domain of quantum mechanics, the FrFT provides an elegant framework for describing wavefunctions and quantum states in phase space. The FrFT is closely related to the Wigner distribution and the linear canonical transform (LCT), making it a natural candidate for analyzing quantum harmonic oscillators, uncertainty principles, and eigenvalue problems. Pioneering works such as Namias (1980) and Qiu et al. (2019) have discussed the role of FrFT in quantum evolution and its connection to the Schrödinger equation, revealing the versatility of the fractional domain in modeling quantum dynamics.

Despite these advances, there remains a gap in the literature concerning the use of the offset fractional Fourier transform (OFrFT) in solving partial differential equations (PDEs), particularly those that govern heat conduction, wave propagation, and diffusion processes. While several studies have applied the classical fractional Fourier transform (FrFT) and linear canonical transform (LCT) to address such equations (Bahri & Ashino, 2020; Namias, 1980; Prasad et al., 2014), the explicit role of the OFrFT in this context remains underexplored. The OFrFT, by incorporating spatial or temporal shifts along with rotation in the time-frequency plane, introduces an additional level of adaptability that can be leveraged to solve generalized forms of classical PDEs. The transformation allows for more flexible boundary modeling and can better accommodate inhomogeneities or nonstationarities in the medium. This paper seeks to fill that gap by exploring the utility of the OFrFT in solving a class of generalized heat equations, thereby extending the analytical toolkit for PDE-based modeling in complex systems.

The heat equation is one of the most fundamental PDEs in mathematical physics, modeling the diffusion of heat in a given medium. Its classical solution via the Fourier transform is well-known; however, when the system exhibits nonstandard boundary conditions, variable diffusivity, or nonhomogeneous media, the standard Fourier framework may become insufficient. In such scenarios, the OFrFT offers a promising alternative. Prior efforts in this direction include Prasad et al. (2014), where the fractional Fourier transform was employed to solve the wave equation, generalizing classical results and demonstrating the advantages of fractional-domain analysis. Similarly, the work of Bahri and Ashino (2020) extended these ideas to both the heat and wave equations through the linear canonical transform, highlighting the theoretical richness and practical benefits of generalized transforms.

Motivated by these developments, the primary goal of this study is to provide a rigorous analytical framework for solving the generalized heat equation using the offset fractional Fourier transform. Our contributions are threefold. First, we establish the mathematical foundation of the OFrFT, including its definition, key properties, and operational rules. Particular emphasis is placed on the convolution theorem, which serves as a vital tool for transforming and solving differential equations. We demonstrate that the convolution theorem in the OFrFT domain can be viewed as a generalization of its classical Fourier counterpart, enabling a broader class of functions and boundary conditions to be handled.

Second, we formulate and solve the generalized heat equation within the OFrFT framework. We begin by applying the transform to the differential operator and deriving the transformed equation in the offset fractional domain. Utilizing the properties of the OFrFT, we obtain an analytical solution that accounts for the offset parameters, thus providing more accurate modeling in systems where standard assumptions are violated. Our approach offers a pathway for addressing initial and boundary value problems in heterogeneous media, where classical methods may falter.

Third, to illustrate the practical implications and validate the effectiveness of the proposed method, we present a detailed example. The example involves a generalized heat conduction problem with spatially varying coefficients and nonhomogeneous boundary conditions. We



solve this problem analytically using the OFrFT framework and compare our results with those obtained through conventional methods. The comparison reveals that the OFrFT-based solution not only aligns with classical results under limiting conditions but also captures additional nuances introduced by the offset parameters.

The implications of this research are far-reaching. Beyond heat conduction, the techniques developed herein can be extended to other types of PDEs, such as the Schrödinger equation, the diffusion equation, and even systems of coupled equations arising in fluid dynamics, electromagnetics, and financial mathematics. Moreover, the OFrFT may serve as a foundation for developing numerical schemes that incorporate transform-domain filtering, regularization, and spectral analysis.

This study also contributes to the theoretical advancement of transform analysis by deepening our understanding of the relationship between the classical Fourier transform, the FrFT, and the OFrFT. Through the derivation of operational properties and their application to PDEs, we offer new insights into how generalized transforms can be systematically employed to address complex problems in applied science and engineering.

In summary, this paper is organized as follows. Section B introduces the necessary mathematical preliminaries and notations used throughout the paper. The definition of the offset fractional Fourier transform and its useful properties is provided in Section C. In Section D, we formulate and solve the generalized heat equation under the OFrFT domain. We apply the transform to the differential operator and utilize the properties of OFrFT to obtain an analytical solution that includes the offset parameters. Finally, Section E concludes the paper by summarizing our main findings and suggesting potential avenues for future research.

B. Notations

First, we outline several notations and lemmas that will serve as a foundation for the rest of this article.

Definition 1. Where $1 \leq p < \infty$, $L^p(\mathbb{R})$ denotes the Banach space of measurable functions on $\mathbb{R}$ furnished with the norm

$$\|f\|_{L^p(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty. \quad (1)$$

More specifically, $L^p(\mathbb{R})$ becomes a Hilbert space with the standard inner product.

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x) \overline{g(x)} dx. \quad (2)$$

We now revisit the definition of the Fourier transform (FT) and its associated lemmas.

Definition 2. The Fourier transform for a function f belonging to $L^1(\mathbb{R})$ is expressed by

$$\hat{f}(\eta) = \mathcal{F}\{f\}(\eta) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\eta x} f(x) dx, \quad \eta \in \mathbb{R}, \quad (3)$$

and for all $f, \hat{f} \in L^1(\mathbb{R})$, the inversion formula is calculated as

$$f(x) = \mathcal{F}^{-1}\{\hat{f}(\eta)\}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\eta x} \hat{f}(\eta) d\eta, \quad x \in \mathbb{R}. \quad (4)$$

Lemma 1. The Fourier transform corresponding to a Gaussian function is

$$\mathcal{F}\{e^{-\alpha x^2}\}(\eta) = \frac{1}{\sqrt{2\alpha}} e^{-\frac{\eta^2}{4\alpha}}, \quad (5)$$

with $\alpha > 0$.

Lemma 2. The Fourier transform corresponding to the Poisson kernel is



$$\mathcal{F}\left\{\sqrt{\frac{2}{\pi}}\frac{y}{x^2+y^2}\right\}(\eta) = e^{-y|\eta|}. \quad (6)$$

Definition 3. Suppose $f \in L^1(\mathbb{R})$. Translation, modulation and dilation operators of the function f are formulated as follows

$$T_a f(x) = f(x-a), \quad M_b f(x) = e^{ibx} f(x), \quad D_c f(x) = \frac{1}{\sqrt{|c|}} f\left(\frac{x}{c}\right), \quad (7)$$

where a, b, c are real constants.

Definition 4. Let $f \in L^1(\mathbb{R})$, then the convolution of the functions f and g , symbolized by $f * g$ is defined as

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)g(x-t)dt, \quad (8)$$

and

$$\mathcal{F}\{f * g\}(\eta) = \mathcal{F}\{f\}(\eta)\mathcal{F}\{g\}(\eta). \quad (9)$$

C. Offset Fractional Fourier Transform and Properties

Definition 5. For $f \in L^1(\mathbb{R})$, the offset fractional Fourier transform (OFrFT) of the function f is given by

$$\mathcal{F}_{\mathbb{O}}^{(\alpha,m,n)}\{f\}(\eta) = \int_{\mathbb{R}} f(t)\mathcal{K}_{(\alpha,m,n)}(t,\eta)dt, \quad (10)$$

with the kernel function $\mathcal{K}_{(\alpha,m,n)}(t,\eta)$ takes the form

$$\mathcal{K}_{(\alpha,m,n)}(t,\eta) = A_{\alpha} e^{\frac{i}{2}m^2 \cot \alpha} e^{\frac{i}{2}((\eta^2+t^2) \cot \alpha + 2t(m-\eta) \csc \alpha + 2\eta(n-m \cot \alpha))}. \quad (11)$$

At this point, parameters (α, m, n) are real numbers and $\alpha \neq n\pi, n \in \mathbb{Z}$.

Theorem 1. For all $f \in L^1(\mathbb{R})$ and $\mathcal{F}_{\mathbb{O}}^{(\alpha,m,n)}\{f\} \in L^1(\mathbb{R})$, the inverse of the offset fractional Fourier transform is defined by

$$f(t) = \int_{\mathbb{R}} \mathcal{F}_{\mathbb{O}}^{(\alpha,m,n)}\{f\}(\eta) \overline{\mathcal{K}_{(\alpha,m,n)}(t,\eta)} d\eta. \quad (12)$$

Theorem 2. Consider that $f(t)$ to be continuously differentiable. If we take as an assumption that

$$\lim_{|t| \rightarrow \infty} f(t) = 0, \quad (13)$$

then

$$\mathcal{F}_{\mathbb{O}}^{(\alpha,m,n)}\left\{\frac{d^n f}{dt^n}\right\}(\eta) = \left(i(\eta - m - n \cot \beta) \sin \beta + \cos \beta \frac{d}{d\eta}\right)^n \mathcal{F}_{\mathbb{O}}^{(\alpha,m,n)}\{f\}(\eta). \quad (14)$$

Theorem 3. Suppose $\mathcal{K}_{(\beta,m,n)}$ be the kernel of the OFrFT. We express $\bar{D}^n$ as follows

$$\bar{D}^n = \left(\frac{\partial}{\partial t} - i(t \cot \beta + m \csc \beta)^n\right), \quad (15)$$

Thus, for all $n \in \mathbb{N}$, it holds that

- i. $\bar{D}^n \mathcal{K}_{(\beta,m,n)} = (-i\eta \csc \beta)^n \mathcal{K}_{(\beta,m,n)}$,
- ii. $\int_{\mathbb{R}} \bar{D}^n \mathcal{K}_{(\beta,m,n)} dt = \int_{\mathbb{R}} \mathcal{K}_{(\beta,m,n)} \bar{D}^n f(t) dt$,
- iii. $\mathcal{F}_{\mathbb{O}}^{(\alpha,m,n)}\{\bar{D}^n f(t)\}(\eta) = (i\eta \csc \beta)^n \mathcal{F}_{\mathbb{O}}^{(\alpha,m,n)}\{f\}(\eta)$,

with $\bar{D}^n = \frac{\partial}{\partial t} - i(t \cot \beta + m \csc \beta)^n$.



D. Offset Fractional Fourier Transform for Generalized Heat Equation

In the following sections, we apply the offset fractional Fourier transform (OFrFT) to address generalized heat equations. We begin by expressing the one-dimensional heat equation in the OFrFT domain.

$$\frac{\partial u(x, t)}{\partial t} = c^2 (\bar{D}_x^*)^2 u(x, t), \quad x \in \mathbb{R}, t \in (0, \infty). \quad (16)$$

In this setting, the initial condition $u(x, 0) = f(x)$ belongs to $L^1(\mathbb{R})$, and $\bar{D}_x^*$ is specified by equation (15) where c represents an arbitrary constant.

Applying the FrFT with respect to x on both sides of equation (16) yields

$$\int_{\mathbb{R}} \mathcal{K}_{(\beta, m, n)}(x, \eta) \frac{\partial u(x, t)}{\partial t} dx = c^2 \int_{\mathbb{R}} \mathcal{K}_{(\beta, m, n)}(x, \eta) (\bar{D}_x^*)^2 u(x, t) dx. \quad (17)$$

One can rewrite this equation as

$$\frac{\partial u_0^{(\beta, m, n)}(x, t)}{\partial t} = -c^2 \eta^2 \csc^2 \beta u_0^{(\beta, m, n)}(x, t). \quad (18)$$

It follows that

$$u_0^{(\beta, m, n)}(\eta, t) = C e^{-c^2 \eta^2 \csc^2 \beta t}, \quad (19)$$

with C denotes an arbitrary constant.

Considering the initial condition $u(x, 0) = f(x)$, it can be shown that

$$u_0^{(\beta, m, n)}(x, 0) = \int_{\mathbb{R}} \mathcal{K}_{(\beta, m, n)}(x, \eta) f(x) dx = C. \quad (20)$$

Replacing (20) in (19) gives

$$u_0^{(\beta, m, n)}(x, t) = \left(\int_{\mathbb{R}} \mathcal{K}_{(\beta, m, n)}(x, \eta) f(x) dx \right) e^{-c^2 \eta^2 \csc^2 \beta t}. \quad (21)$$

Taking the inverse of the FrFT in (21), we see that

$$u(x, t) = \left(\mathcal{F}_{\mathbb{O}}^{(\beta, m, n)} \right)^{-1} \left\{ u_0^{(\beta, m, n)}(x, t) \right\} (\eta, t). \quad (22)$$

As a consequence of equation (12), we additionally obtain

$$\begin{aligned} u(x, t) &= \left(\mathcal{F}_{\mathbb{O}}^{(\beta, m, n)} \right)^{-1} \left\{ u_0^{(\beta, m, n)}(x, t) \right\} (\eta, t) \\ &= \overline{A_{\beta}} \int_{\mathbb{R}} e^{-\frac{i}{2}((\eta^2 + x^2 + m^2) \cot \beta + 2x(m - \eta) \csc \beta + 2\eta(n - m \cot \beta))} \\ &\quad \times (A_{\beta}) \int_{\mathbb{R}} e^{\frac{i}{2}((\eta^2 + x^2 + m^2) \cot \beta + 2x(m - \eta) \csc \beta + 2\eta(n - m \cot \beta))} f(x) dx e^{-c^2 \eta^2 \csc^2 \beta t} d\eta \\ &= |A_{\beta}|^2 \int_{\mathbb{R}} e^{-\frac{i}{2}(x^2 \cot \beta + 2x(m - \eta) \csc \beta)} \left(\int_{\mathbb{R}} e^{\frac{i}{2}(x^2 \cot \beta + 2x(m - \eta) \csc \beta)} f(x) dx \right) e^{-c^2 \eta^2 \csc^2 \beta t} d\eta \\ &= |A_{\beta}|^2 \int_{\mathbb{R}} e^{-\frac{i}{2}x^2 \cot \beta - ixm \csc \beta} \\ &\quad \times \int_{\mathbb{R}} e^{ixm \csc \beta} \left(\int_{\mathbb{R}} e^{\frac{ix^2 \cot \beta}{2} + ixm \cot \beta} e^{-ix\eta \csc \beta} f(x) dx \right) e^{-c^2 \eta^2 \csc^2 \beta t} d\eta. \end{aligned} \quad (23)$$

Set

$$g^{(\beta, m)}(x) = e^{\frac{ix^2 \cot \beta}{2} + ixm \cot \beta} f(x), \quad (24)$$

and

$$\eta \csc \beta = v. \quad (25)$$

Equation (23) above implies that

$u(x, t)$



$$\begin{aligned}
 &= |A_\beta|^2 e^{-\frac{i}{2}x^2 \cot \beta - ixm \csc \beta} \sin \beta \int_{\mathbb{R}} e^{ixv} \left(\int_{\mathbb{R}} g^{(\beta,m)}(x) e^{-ixv} dx \right) e^{-c^2 v^2 t} dv \\
 &= |A_\beta|^2 e^{-\frac{i}{2}x^2 \cot \beta - ixm \csc \beta} \sin \beta \sqrt{2\pi} \int_{\mathbb{R}} e^{ixv} e^{-c^2 v^2 t} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g^{(\beta,m)}(x) e^{-ixv} dx \right) dv \\
 &= \sqrt{2\pi} \sin \beta |A_\beta|^2 e^{-\frac{i}{2}x^2 \cot \beta - ixm \csc \beta} \int_{\mathbb{R}} e^{ixv} e^{-c^2 v^2 t} \mathcal{F}\{g^{(\beta,m)}\}(x) dv. \tag{26}
 \end{aligned}$$

In view of equation (5), we have

$$\mathcal{F}\left\{\frac{1}{\sqrt{2c^2 t}} e^{-\frac{x^2}{4c^2 t}}\right\}(v) = e^{-c^2 v^2 t}. \tag{27}$$

Inserting equation (27) into equation (26) yields

$$\begin{aligned}
 u(x, t) &= \sqrt{2\pi} \sin \beta |A_\beta|^2 e^{-\frac{i}{2}x^2 \cot \beta - ixm \csc \beta} \int_{\mathbb{R}} e^{ixv} \mathcal{F}\left\{\frac{1}{\sqrt{2c^2 t}} e^{-\frac{x^2}{4c^2 t}}\right\}(v) \mathcal{F}\{g^{(\beta,m)}\}(x) dv \\
 &= \frac{\sqrt{2\pi} \sin \beta |A_\beta|^2}{\sqrt{2c^2 t}} e^{-\frac{i}{2}x^2 \cot \beta - ixm \csc \beta} \int_{\mathbb{R}} \mathcal{F}\left\{e^{-\frac{x^2}{4c^2 t}}\right\}(v) \mathcal{F}\{g^{(\beta,m)}\}(v) e^{ixv} dv. \tag{28}
 \end{aligned}$$

Applying equation (9) to equation (28) leads to

$$\begin{aligned}
 u(x, t) &= \frac{\sqrt{2\pi} \sin \beta |A_\beta|^2}{\sqrt{2c^2 t}} e^{-\frac{i}{2}x^2 \cot \beta - ixm \csc \beta} \left(\mathcal{F}_{\mathbb{O}}^{(\beta,m,n)} \right)^{-1} \left\{ \mathcal{F}\left\{e^{-\frac{x^2}{4c^2 t}}\right\}(v) \mathcal{F}\{g^{(\beta,m)}\}(v) \right\}(x, t) \\
 &= \frac{\sqrt{2\pi} \sin \beta |A_\beta|^2}{\sqrt{2c^2 t}} e^{-\frac{i}{2}x^2 \cot \beta - ixm \csc \beta} \left(e^{-\frac{x^2}{4c^2 t}} * g^{(\beta,m)}(x) \right)(y) \\
 &= \frac{\sqrt{2\pi} \sin \beta |A_\beta|^2}{\sqrt{2c^2 t}} e^{-\frac{i}{2}x^2 \cot \beta - ixm \csc \beta} \left(e^{-\frac{x^2}{4c^2 t}} * e^{\frac{ix^2 \cot \beta}{2} + ixm \csc \beta} f(x) \right)(y) \\
 &= \frac{\sqrt{2\pi} \sin \beta |A_\beta|^2}{\sqrt{2c^2 t}} e^{-\frac{i}{2}x^2 \cot \beta - ixm \csc \beta} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4c^2 t}} e^{\frac{iy^2 \cot \beta}{2} + iym \csc \beta} f(y) dy. \tag{29}
 \end{aligned}$$

Specifically, if $\beta = \frac{\pi}{2}$, relation (29) simplifies to

$$u(x, t) = \frac{1}{\sqrt{4\pi c^2 t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4c^2 t}} f(y) dy, \tag{30}$$

representing the solution to the heat equation through the classical Fourier transform.

When $m = 0$, relation (29) above reduce to

$$u(x, t) = \frac{e^{\frac{ix^2 \cot \beta}{2}}}{\sqrt{4\pi c^2 t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4c^2 t}} e^{\frac{iy^2 \cot \beta}{2}} f(y) dy, \tag{31}$$

which is the solution of the heat equation using the classical Fourier transform.

An example is presented below to support and illustrate the preceding result.

Example 1. Determine the solution $u(x, t)$ of equation (29) with $c = 1$ and

$$f(x) = \begin{cases} 1, & |x| < 2 \\ 0, & \text{otherwise.} \end{cases} \tag{32}$$

Solution. Using (32) together with (29), we arrive at

$$\begin{aligned}
 u(x, t) &= \frac{\sqrt{2\pi} \sin \beta |A_\beta|^2}{\sqrt{2t}} e^{-\frac{i}{2}x^2 \cot \beta - ixm \csc \beta} \int_{-2}^2 e^{-\frac{(x-y)^2}{4t}} e^{\frac{iy^2 \cot \beta}{2} + iym \csc \beta} dy
 \end{aligned}$$



$$\begin{aligned}
 &= \frac{\sqrt{2\pi} \sin \beta |A_\beta|^2}{\sqrt{2t}} e^{-\frac{i}{2}x^2 \cot \beta - ixm \csc \beta} \int_{-2}^2 e^{-\frac{(x^2-y^2-2xy)^2}{4t}} e^{\frac{i}{2}y^2 \cot \beta + iym \csc \beta} dy \\
 &= \frac{\sqrt{2\pi} \sin \beta |A_\beta|^2}{\sqrt{2t}} e^{-\frac{i}{2}x^2 \cot \beta - ixm \csc \beta - \frac{x^2}{4t}} \int_{-2}^2 e^{-\left(\frac{1}{4t} - \frac{i}{2} \cot \beta\right)y^2 + \left(\frac{2x}{4t} + im \csc \beta\right)y} dy. \quad (33)
 \end{aligned}$$

Equation (33) may be reformulated as

$u(x, t)$

$$\begin{aligned}
 &= \frac{\sqrt{2\pi} \sin \beta |A_\beta|^2}{\sqrt{2t}} e^{-\frac{i}{2}x^2 \cot \beta - ixm \csc \beta - \frac{x^2}{4t}} \int_{-2}^2 e^{-\left(\frac{1}{4t} - \frac{i}{2} \cot \beta\right)\left(y^2 - \frac{\left(\frac{2x}{4t} + im \csc \beta\right)y}{\left(\frac{1}{4t} - \frac{i}{2} \cot \beta\right)}\right)} dy \\
 &= \frac{\sqrt{2\pi} \sin \beta |A_\beta|^2}{\sqrt{2t}} e^{-\frac{i}{2}x^2 \cot \beta - ixm \csc \beta - \frac{x^2}{4t}} \int_{-2}^2 e^{-\left(\frac{1}{4t} - \frac{i}{2} \cot \beta\right)\left(y^2 - \frac{(2x + i4tm \csc \beta)y}{1 - i2t \cot \beta}\right)} dy. \quad (34)
 \end{aligned}$$

The preceding equation yields

$$\begin{aligned}
 u(x, t) &= \frac{\sqrt{2\pi} \sin \beta |A_\beta|^2}{\sqrt{2t}} e^{-\frac{i}{2}x^2 \cot \beta - ixm \csc \beta - \frac{x^2}{4t}} \\
 &\quad \times \int_{-2}^2 e^{-\left(\frac{1}{4t} - \frac{i}{2} \cot \beta\right)\left(y - \frac{1}{2}\left(\frac{2x + i4tm \csc \beta}{1 - i2t \cot \beta}\right)\right)^2 - \frac{1}{4}\left(\frac{2x + i4tm \csc \beta}{1 - i2t \cot \beta}\right)^2} dy \\
 &= \frac{\sqrt{2\pi} \sin \beta |A_\beta|^2}{\sqrt{2t}} e^{-\frac{i}{2}x^2 \cot \beta - ixm \csc \beta - \frac{x^2}{4t}} \\
 &\quad \times \int_{-2}^2 e^{-\left(\frac{1}{4t} - \frac{i}{2} \cot \beta\right)\left(y - \frac{1}{2}\left(\frac{2x + i4tm \csc \beta}{1 - i2t \cot \beta}\right)\right)^2} e^{\frac{1}{16t} \frac{(2x + i4tm \csc \beta)^2}{1 - i2t \cot \beta}} dy \\
 &= \frac{\sqrt{2\pi} \sin \beta |A_\beta|^2}{\sqrt{2t}} e^{-\frac{i}{2}x^2 \cot \beta - ixm \csc \beta - \frac{x^2}{4t} + \frac{1}{16t} \frac{(2x + i4tm \csc \beta)^2}{1 - i2t \cot \beta}} \\
 &\quad \times \int_{-2}^2 e^{-\left(\frac{1}{2}\left(\frac{2x + i4tm \csc \beta}{1 - i2t \cot \beta}\right)\sqrt{\frac{1}{4t} - \frac{i}{2} \cot \beta} - \sqrt{\frac{1}{4t} - \frac{i}{2} \cot \beta} y\right)^2} dy. \quad (35)
 \end{aligned}$$

Let

$$v = \frac{1}{2} \sqrt{\frac{1}{4t} - \frac{i}{2} \cot \beta} \left(\frac{2x + i4tm \csc \beta}{1 - i2t \cot \beta} \right) - \sqrt{\frac{1}{4t} - \frac{i}{2} \cot \beta} y, \quad (36)$$

then,

$$\begin{aligned}
 u(x, t) &= \frac{\sqrt{2\pi} \sin \beta |A_\beta|^2}{\sqrt{2t}} e^{-\frac{i}{2}x^2 \cot \beta - ixm \csc \beta - \frac{x^2}{4t} + \frac{1}{16t} \frac{(2x + i4tm \csc \beta)^2}{1 - i2t \cot \beta}} \\
 &\quad \times \int_{\frac{1}{2}\sqrt{\frac{1}{4t} - \frac{i}{2} \cot \beta} \left(\frac{2x + i4tm \csc \beta}{1 - i2t \cot \beta}\right) - 2\sqrt{\frac{1}{4t} - \frac{i}{2} \cot \beta}}^{\frac{1}{2}\sqrt{\frac{1}{4t} - \frac{i}{2} \cot \beta} \left(\frac{2x + i4tm \csc \beta}{1 - i2t \cot \beta}\right) + 2\sqrt{\frac{1}{4t} - \frac{i}{2} \cot \beta}} e^{-v^2} \frac{dv}{\sqrt{\frac{1}{4t} - \frac{i}{2} \cot \beta}} \\
 &= \frac{\sqrt{2\pi} \sin \beta |A_\beta|^2}{\sqrt{2t} \sqrt{\frac{1}{4t} - \frac{i}{2} \cot \beta}} e^{-\frac{i}{2}x^2 \cot \beta - ixm \csc \beta - \frac{x^2}{4t} + \frac{1}{16t} \frac{(2x + i4tm \csc \beta)^2}{1 - i2t \cot \beta}} \\
 &\quad \times \left(\frac{\pi}{2} \left(\operatorname{erf} \left(\frac{1}{2} \sqrt{\frac{1}{4t} - \frac{i}{2} \cot \beta} \left(\frac{2x + i4tm \csc \beta}{1 - i2t \cot \beta} \right) - 2 \sqrt{\frac{1}{4t} - \frac{i}{2} \cot \beta} \right) \right) - \operatorname{erf} \left(\frac{1}{2} \sqrt{\frac{1}{4t} - \frac{i}{2} \cot \beta} \left(\frac{2x + i4tm \csc \beta}{1 - i2t \cot \beta} \right) + 2 \sqrt{\frac{1}{4t} - \frac{i}{2} \cot \beta} \right) \right).
 \end{aligned}$$

As a result,

$$u(x, t) = \frac{2\sqrt{2}}{\sqrt{1 - i2t \cot \beta}} e^{-\frac{i}{2}x^2 \cot \beta - i x m \csc \beta - \frac{x^2}{4t} + \frac{1}{16t} \frac{(2x + i4tm \csc \beta)^2}{1 - i2t \cot \beta}} \times \left(\operatorname{erf} \left(\frac{1}{2} \sqrt{\frac{1}{4t} - \frac{i}{2} \cot \beta} \left(\frac{2x + i4tm \csc \beta}{1 - i2t \cot \beta} \right) - 2 \sqrt{\frac{1}{4t} - \frac{i}{2} \cot \beta} \right) - \operatorname{erf} \left(\frac{1}{2} \sqrt{\frac{1}{4t} - \frac{i}{2} \cot \beta} \left(\frac{2x + i4tm \csc \beta}{1 - i2t \cot \beta} \right) + 2 \sqrt{\frac{1}{4t} - \frac{i}{2} \cot \beta} \right) \right). \quad (37)$$

At this point,

$$\operatorname{erf}(\eta) = \frac{2}{\pi} \int_0^\eta e^{-z^2} dz, \quad (38)$$

holds for all values of η . The simulation of equation (37) with $c = 1$ and under varying β and m are summarized in Table 1.

Table 1 The solutions obtained for Example 1 for $c = 1$ and different values of β and m .

β	c	m	$u(x, t)$
$\frac{\pi}{2}$	1	0	$\frac{1}{2} \left(\operatorname{erf} \left(\frac{x-2}{2\sqrt{t}} \right) - \operatorname{erf} \left(\frac{x+2}{2\sqrt{t}} \right) \right)$
$\frac{\pi}{3}$	1	2	$\frac{2\sqrt{2}}{\sqrt{\frac{\sqrt{3}-i2t}{\sqrt{3}}}} e^{-\left(\frac{i}{2\sqrt{3}} + \frac{1}{4t} - \frac{\sqrt{3}}{4t(\sqrt{3}-i2t)}\right)x^2 + \left(\frac{i4t}{\sqrt{3}-i2t} - \frac{i4}{\sqrt{3}}\right)x - \frac{16\sqrt{3}t}{3(\sqrt{3}-i2t)}} \times \left(\operatorname{erf} \left(\frac{1}{2} \sqrt{\frac{\sqrt{3}-i2t}{4t\sqrt{3}}} \left(\frac{2\sqrt{3} + i16t}{\sqrt{3}-i2t} \right) - 2 \sqrt{\frac{\sqrt{3}-i2t}{4t\sqrt{3}}} \right) - \operatorname{erf} \left(\frac{1}{2} \sqrt{\frac{\sqrt{3}-i2t}{4t\sqrt{3}}} \left(\frac{2\sqrt{3} + i16t}{\sqrt{3}-i2t} \right) + 2 \sqrt{\frac{\sqrt{3}-i2t}{4t\sqrt{3}}} \right) \right)$
$\frac{\pi}{4}$	1	2	$\frac{2\sqrt{2}}{\sqrt{1-2t}} e^{-\left(\frac{i}{2} + \frac{1}{4t} - \frac{1}{4t(1-i2t)}\right)x^2 + \left(\frac{i2t}{1-i2t} - 2i\right)x - \frac{4t}{1-i2t}} \times \left(\operatorname{erf} \left(\frac{1}{2} \sqrt{\frac{1}{4t} - \frac{i}{2} \left(\frac{2x + i8t}{i - i2t} \right)} - 2 \sqrt{\frac{1}{4t} - \frac{i}{2}} \right) - \operatorname{erf} \left(\frac{1}{2} \sqrt{\frac{1}{4t} - \frac{i}{2} \left(\frac{2x + i8t}{i - i2t} \right)} + 2 \sqrt{\frac{1}{4t} - \frac{i}{2}} \right) \right)$

As shown in Table 1, for $\beta = \frac{\pi}{2}$ and $m = 0$, equation (37) simplifies to

$$u(x, t) = \frac{1}{2} \left(\operatorname{erf} \left(\frac{x-2}{2\sqrt{t}} \right) - \operatorname{erf} \left(\frac{x+2}{2\sqrt{t}} \right) \right), \quad (39)$$

which closely resembles to the solution of the classical heat equation using the Fourier transform as illustrated in Figure 1. Figures 2 and 3 present the solution of Example 1 with $m = 2$ and several time values at each specified value of β .

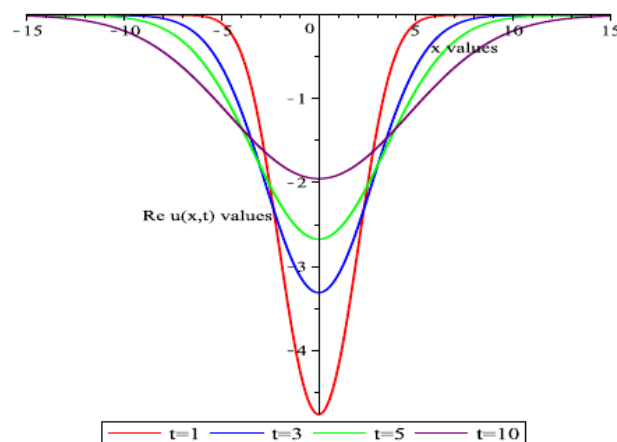


Figure 1 Real part of solution of Example 1 for $m = 0$, $\beta = \frac{\pi}{2}$, and $t = 1, 3, 5, 10$.

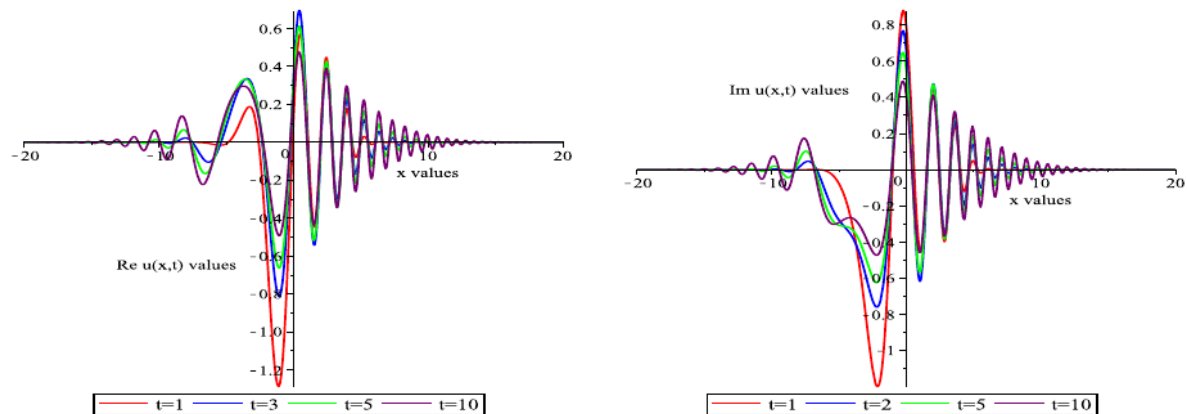


Figure 2 Real and imaginary parts of solution of Example 1 for $m = 2$, $\beta = \frac{\pi}{3}$, and $t = 1, 3, 5, 10$.

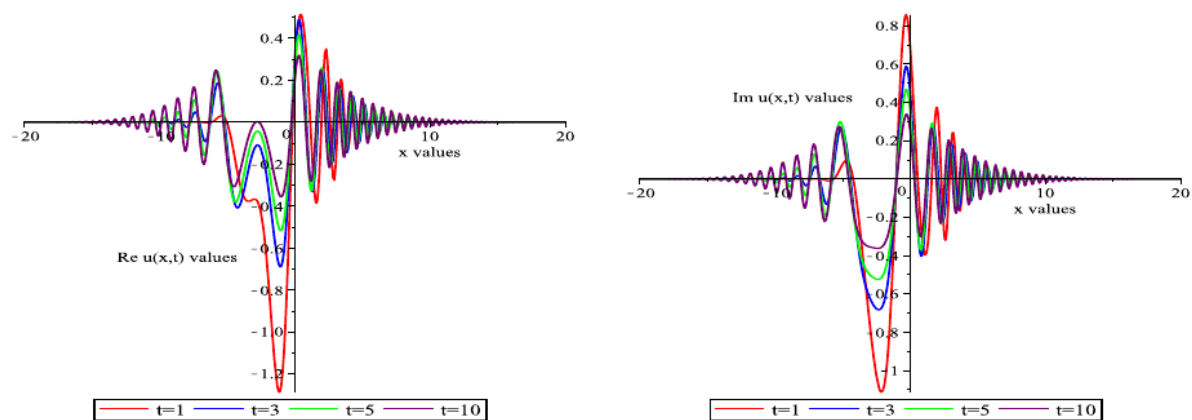


Figure 2 Real and imaginary parts of solution of Example 1 for $m = 2$, $\beta = \frac{\pi}{4}$, and $t = 1, 3, 5, 10$.

E. Conclusion

The offset fractional Fourier transform has been applied in this work to derive the solution of the generalized heat equation. By applying the transform to the differential operator and leveraging the properties of OFrFT, we derive an analytical solution that incorporates the offset parameters. This study demonstrates the potential of OFrFT in providing a more flexible analytical framework for solving partial differential equations beyond the limitations of the standard Fourier transform. Future research may focus on a deeper investigation of the properties and uncertainty principles of OFrFT, as well as extend its application to explore solutions of other equations, such as the Laplace equation, within the OFrFT framework.

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