

ON INVOLUTE AND EVOLUTE OF PSEUDONULL AND NULL CURVES IN MINKOWSKI 3-SPACES

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Abstract. The aim of this paper is to discuss about the theory of involute and evolute of pseudo-null and null curves in Minkowski space. We apply the Frenet equations of the pseudo-null and null curves in Minkowski 3-space to derive their involutes and the evolutes. Furthermore, some properties of the involutes and evolutes of pseudo-null and null curves in Minkowski 3-space are shown.

Keywords: PseudoNull curves, Null Curves, Involutives, Evolutes, Frenet Equations.

A. Introduction

Theory of curves in Minkowski space is very interesting for mathematics and physics. One of the most important application of theory of curves in Minkowski spaces is related to special and general relativity theory (Formiga & Romero, 2008). Because metric in Minkowski space is different from the metric in Euclidean space, therefore some problems become a little strange and different. However, theories of curve in Minkowski space are generally developed by using the properties of curve in Euclidean space but in different metric.

Minkowski space is the metric space $\mathbb{R}_1^3$ defined by Minkowski metric

$$\langle u, v \rangle = -u_1v_1 + u_2v_2 + u_3v_3$$

for all $u, v \in \mathbb{R}$. In this paper, the Minkowski space will be denoted by E_1^3 .

The norm of $u \in E_1^3$ is defined by $\|u\| = \sqrt{|\langle u, u \rangle|}$. A vector $u \in E_1^3$ is called spacelike if $\langle u, u \rangle > 0$, timelike $\langle u, u \rangle < 0$, and null or lightlike if $\langle u, u \rangle = 0, u \neq 0$. The Lorentzian vector product of u and v in E_1^3 is defined by

$$u \times v = (-u_2v_3 + u_3v_2, u_1v_3 - u_3v_1, u_1v_2 - u_2v_1).$$

A curve $\alpha: I \rightarrow E_1^3$ is said spacelike (resp. timelike, null) at $t \in I$ if $\alpha'(t)$ is a spacelike (resp. timelike, null) vector. In case $\alpha'(t)$ is a spacelike (resp. timelike, null) vector for all $t \in I$, then α is a spacelike (resp. timelike, null). For non null curves, the arc-length is defined as $s = \sqrt{|\langle \alpha', \alpha' \rangle|}$ while for null curves, the pseudo arc-length is defined as $s = \sqrt{|\langle \alpha'', \alpha'' \rangle|}$. If $|\langle \alpha', \alpha' \rangle| = 1$, the non null curve is said parametrized by arc-length, and if $|\langle \alpha', \alpha' \rangle| = 1$, the null curve is said parametrized by pseudo arc-length. Theory of curves in Minkowski space has been studied by many mathematicians such as Sasaki (2010); Ferrandez, Gimenez & Lucas (2002); Lopez (2014); and Inoguchi & Lee (2008). Furthermore, the applications of the curves in Minkowski space have been studied by Duggal (1996, 2007), Mohajan (2013), and O'Neill (1983).

The theory of involute and evolute was firstly introduced by C. Huygens in 1673 in 1673 when he tried to build an accurate clock called isochronous pendulum clock. He found that isochronous curve is an arc of cycloid and that involute of cycloid is a similar cycloid (Merzach and Boyer, 2010). In classical differential geometry, a curve α^* is called an involute of α if



curve α^* is lying in the tangent surface of α and their tangent lines are perpendicular in all point on the curves. Conversely, α is called the evolute of α^* . This concept can be applied to the curve in semi-Riemann geometry, especially in Minkowski space. The theory of involute-evolute of curves in Minkowski has been studied by many mathematicians such as Bukcu and Karacan (2009, 2007a, 2007b), Almaz & Kulahci (2018) and Hanif, Hou and Nesovic (2019).

In this paper, we introduce the different kind of curves in Minkowski space based on the causality character of their tangent vectors and their Frenet equations. In the next section we introduce the concept of Involute of curves in Minkowski space. In this section the discussion will be discussed separately based on the type of curves in Minkowski space. In the last section the evolute of the different curves in Minkowski space are provided with some their properties.

B. Preliminaries

Let $\{T(s), N(s), B(s)\}$ be a Frenet trihedron in 3-dimensional Minkowski space. Assume tha the curve is parametrized by the arc-length or pseudo arc-length s and $T(s)$ is the tangent vector of the curve $\alpha(s)$. Unlike those in Euclidean space, curve in Minkowski space have some special cases as follows:

1. if the curve is null, then $B(s)$ is also null which implies $\{T(s), N(s), B(s)\}$ is not orthogonal basis. In this case, we will aply null frame for finding the Frenet equations of the curves.
2. if the curves are non null curves, we assume $\{T(s), N(s), B(s)\}$ is orthonormal basis in E_1^3 . The binormal vector $B(s)$ will be always defined as the cross product between tangent vector $T(s)$ and normal vector $N(s)$. Moreover, the basis $\{T(s), N(s), B(s)\}$ can be either positively or negatively oriented. The basis is positively oriented if $B(s)$ is a timelike vector and negatively oriented if $B(s)$ is a spacelike vector.
3. it will be desirable that in the case $\{T(s), N(s), B(s)\}$ is an orthonormal basis, this basis is future-directed. This cannot assume priori. Even in the case that the curve is a timelike, $T(s)$ could not be future-oriented.

Now, let $\alpha: I \rightarrow E_1^3$ be a regular curve parametrized by arc-length or pseudo arc-length s , then $T(s)$ is the unit tangent vector at s since $\langle T(s), T(s) \rangle$ is a constant with value $-1, 0$, or 0 depending whether the curve is timelike, spacelike, or null respectively.

Pseudonull curve $\beta(s)$ is a spacelike curve with null principal normal vector. The vector $T'(s)$ is a null for all s . Since we have null vectors then the basis $\{T(s), N(s), B(s)\}$ is not orthonormal basis in E_1^3 . Therefore, to find the Frenet equation, we must apply null frame concept. We define the normal vector as $N(s) = T'(s)$, which is linearly independent with $T(s)$. Let $B(s)$ be the unique lightlike such that $\langle N(s), B(s) \rangle = 1$, and it is orthogonal to $T(s)$. The vector $B(s)$ is binormal vector of β at s . The Frenet equations are

$$\begin{pmatrix} T'(s) \\ N'(s) \\ B'(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \tau & 0 \\ -1 & 0 & -\tau \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} \quad (1)$$

The function τ is called the pseudo-torsion of $\beta(s)$ and it is obtained by $\tau = \langle N'(s), B(s) \rangle$. In the case we do not know if its is positively or negatively oriented.

Let $\alpha(s)$ be a null curve parametrized by pseudo arc-length s . Then the tangent vector $T(s)$ is also a null vector. Since we have null vector as a component of basis, then the basis $\{T(s), N(s), B(s)\}$ is not orthonormal in E_1^3 and we must apply the null frame concept to find the Frenet equations. Define the normal vector $N(s) = T'(s)$ which is a unite spacelike vector.



We also define the binormal vector $B(s)$, the unit null vector which is orthogonal to normal vector $N(s)$ and $\langle T(s), B(s) \rangle = 1$. Therefore, the Frenet equations are

$$\begin{pmatrix} T'(s) \\ N'(s) \\ B'(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \tau & 0 & -1 \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} \quad (1)$$

where the pseudo-torsion of $\alpha(s)$ is $\tau = \langle N'(s), B(s) \rangle$ (Lopez, 2014).

C. Main Results

1. Involutes of Pseudo-Null and Null Curves in Minkowski 3-Space

Definition 1. Let $r: I \rightarrow E_1^3$ and $r^*: I \rightarrow E_1^3$ be curves in Minkowski 3-spaces. For all $s \in I$, $r^*(s)$ is called the involute of $r(s)$ if the tangent vector of r at point $r(s)$ passes through the tangent at the point $r^*(s)$ of the curve $r^*(s)$ and

$$\langle T^*(s), T(s) \rangle = 0 \quad (3)$$

where T^* and T are the tangent vectors of r and r^* , respectively.

Theorem 1. Let $\beta: I \rightarrow E_1^3$ be a pseudonull or null curve parametrized by arc-length s and $\beta^*: I \rightarrow E_1^3$ be its involutes. Let c be a constant real number, then

$$\beta^*(s) = \beta(s) + (c - s)T(s) \quad (4)$$

where $T(s)$ is the tangent vector of $\beta(s)$.

Proof. Let $\beta(s)$ be a pseudonull or null curve in Minkowski 3-spaces. The tangent lines of the curve $\beta(s)$ will construct a tangent surface. $\beta^*(s)$ is the involute of $\beta(s)$ if $\beta^*(s)$ lies on the tangent surface and orthogonal to all tangent line of $\beta(s)$. Let x^* be the point in $\beta^*(s)$ which crosses the tangent line $T(s)$ of $\beta(s)$ at point x . Then, $x^* - x$ is proportional to $T(s)$. As a consequence, $\beta^*(s)$ can be represented in the form of $x^*(s) = x(s) + k(s)T(s)$ for some value of constants depending on s . Therefore, the tangent vector of $\alpha^*(s)$ will be

$$\begin{aligned} \beta^{*'}(s) &= \beta'(s) + k'(s)T(s) + k(s)T'(s) \\ &= (1 + k'(s))T(s) + k(s)\kappa(s)N(s) \end{aligned}$$

By Definition 1, this tangent vector is orthogonal to T which implies

$$0 = \langle \beta^* \cdot T \rangle = (1 + k')\langle T, T \rangle + k\kappa\langle N, T \rangle.$$

It follows that

$$1 + k' = 0. \quad (5)$$

Integrating equation (5) gives $k = -s + c$, for some constant c . As a consequence, there exists an infinite family of involutes and the equation of the involute curves $\beta^*(s)$ is given by equation (4) as desired.

Theorem 2. Let $\beta: I \rightarrow E_1^3$ be a pseudonull or null curve parametrized by arc-length s and $\beta^*: I \rightarrow E_1^3$ be its involute. Then the distance between $\beta(s)$ and $\beta^*(s)$ is $|c - s|$.

Proof. By equation (4), we have

$$\beta^*(s) - \beta(s) = (c - s)T(s).$$

Therefore,

$$\|\beta^*(s) - \beta(s)\| = \|(c - s)T(s)\| = |c - s|\|T(s)\| = |c - s|.$$

Theorem 3. Let $\beta: I \rightarrow E_1^3$ be a pseudonull curve in Minkowski 3-space parametrized by the arc-length s . Then the involute $\beta^*(s)$ of $\beta(s)$ is a null curve.

Proof. Taking derivative of equation (4) and using equation (2) give us

$$\beta^{*'}(s) = T(s) + (c - s)N(s) - T(s) = (c - s)N(s).$$

Since the principal normal vector $N(s)$ is null, then we have



$$\langle \beta^{*'}(s), \beta^{*'}(s) \rangle = 0.$$

Hence $\beta^*(s)$ is a null curve and it completes the proof.

Theorem 4. Let $\alpha(s)$ be a null curve in Minkowski 3-spaces parametrized by the pseudo arc-length s . Then the involute $\alpha^*(s)$ of $\alpha(s)$ is a null curve/

Proof. Differentiating the equation (4) and using equation (2) imply

$$\alpha^{*'}(s) = (c - s)N(s).$$

Consequently,

$$\langle \alpha^{*'}(s), \alpha^{*'}(s) \rangle = (c - s)\langle N(s), N(s) \rangle = 0.$$

Hence $\alpha^*(s)$ is a null curve and it completes the proof.

2. Evolutes of Pseudo-Null and Null Curves in Minkowski 3-Space

Definition 2. Let $r(s)$ be involute of the curve $r^*(s)$ parametrized by arc-length or pseudo arc-length s . Therefore, the tangent line of $r^*(s)$ intersect $r(s)$ orthogonally. If x^* is the point of contact on the evolute to the tangent line which intersects β at $x(s)$, then $x^* - x$ can be represented as a linear combination of normal vector $N(s)$ and binormal vector $B(s)$. Therefore, it can be written as

$$x^*(s) = x(s) = k_1(s)N(s) + k_2(s)B(s). \quad (6)$$

Next, we will find the function $k_1(s)$ and $k_2(s)$ by considering the causal characteristics of the curves.

Theorem 5. Let $\beta^*(s)$ be the evolute of the pseudonull curve $\beta(s)$ parametrized by arc-length s and curvature $\kappa(s) = 1$. Then

$$\beta^*(s) = \beta(s) + e^{-2\psi(s)+c}N(s) + B(s) \quad (7)$$

where $-\psi(s) + \frac{1}{2}c = \int \tau(s) ds$.

Proof. Differentiating equation (6) yields

$$\begin{aligned} \beta^{*'}(s) &= \beta'(s) + k_2'(s)N(s) + k_1(s)(\tau(s)N(s)) + k_2'(s)B(s) \\ &\quad + k_2(s)(-\kappa(s)T(s) - \tau(s)B(s)) \\ &= (1 - k_2(s))T(s) + (k_1'(s) + k_1(s)\tau(s))N(s) + (k_2'(s) - k_2(s)\tau(s))B(s). \end{aligned}$$

As a consequence,

$$\begin{aligned} \langle \beta^{*'}(s), T(s) \rangle &= (1 - k_2(s))\langle T(s), T(s) \rangle + (k_1'(s) + k_1(s)\tau(s))\langle N(s), T(s) \rangle \\ &\quad + (k_2'(s) - k_2(s)\tau(s))\langle B(s), T(s) \rangle. \end{aligned}$$

Since $\langle T^*(s), T^*(s) \rangle = 0$, then we have $0 = 1 - k_2(s)$, and hence $k_2(s) = 1$. Therefore, it implies

$$\beta^{*'}(s) = (k_1'(s) + k_1(s)\tau(s))N(s) - \tau(s)B(s). \quad (8)$$

On the other hand, $\beta^{*'}(s)$ is proportional to $\beta^*(s) - \beta(s) = k_1(s)N(s) + k_2(s)B(s)$. As a consequence,

$$\frac{k_1'(s) + k_1(s)\tau(s)}{k_1(s)} = -\tau(s)$$

which gives us

$$-2\tau(s)k_1(s) = k_1'(s). \quad (9)$$

Solving this differential equation yields

$$k_1(s) = e^{\int -\tau(s)ds} = e^{-2\psi(s)+c}$$

for some constant c . Therefore, substituting $k_1(s)$ and $k_2(s)$ into equation (6) yields equation (7) and it completes the proof.



Theorem 6. Let $\beta: I \rightarrow E_1^3$ be a pseudonull curve parametrized by ar-length (s) and $\beta^*: I \rightarrow E_1^3$ be its evolute. Then the distance between $\beta(s)$ and $\beta^*(s)$ is $\sqrt{2e^{-2\psi(s)+c}}$.

Proof. From equation (7), we have

$$\beta^*(s) - \beta(s) = e^{-2\psi(s)+c}N(s) + B(s).$$

It follows that

$$\begin{aligned}\|\beta^*(s) - \beta(s)\|^2 &= \langle e^{-2\psi(s)+c}N(s) + B(s), e^{-2\psi(s)+c}N(s) + B(s) \rangle \\ &= 2e^{-2\psi(s)+c}\langle N(s), B(s) \rangle \\ &= 2e^{-2\psi(s)+c}.\end{aligned}$$

Therefore,

$$\|\beta^*(s) - \beta(s)\| = \sqrt{2e^{-2\psi(s)+c}}$$

as desired.

Theorem 8. Let $\beta(s)$ be a pseudonull curve parametrized by arc-length s with curvature $\kappa(s) = 1$ and torsion $\tau(s) \neq 0$. Then, the evolute of $\beta(s)$ is a spacelike curve.

Proof. From equations (8) and (9), we have

$$\beta^{*'}(s) = -k_1(s)\tau(s)N(s) - \tau(s)B(s).$$

Therefore,

$$\begin{aligned}\langle \beta^{*'}(s), \beta^{*'}(s) \rangle &= \langle -k_1(s)\tau(s)N(s) - \tau(s)B(s), -k_1(s)\tau(s)N(s) - \tau(s)B(s) \rangle \\ &= k_1(s)\tau^2(s) \\ &= e^{-2\psi(s)+c}\tau^2(s) > 0\end{aligned}$$

Since $N(s)$ and $B(s)$ are null vectors. Therefore $\beta^*(s)$ is a spacelike curve and completes the proof.

Theorem 9. Let $\alpha^*(s)$ be the evolute of a null curve $\alpha(s)$ parametrized by pseudo-arc length s . Then

$$\alpha^*(s) = \alpha(s) - \frac{1}{\tau(s)}N(s) - (\psi(s) + c)B(s) \quad (10)$$

where $\psi(s) + c = \int \frac{1}{\tau(s)} ds$.

Proof. If we take the derivative of equation (10), we have

$$\begin{aligned}\alpha^{*'}(s) &= \alpha'(s) + k_1'(s)N(s) + k_1(s)N'(s) + k_2'(s)B(s) + k_2(s)B'(s) \\ &= T(s) + k_1'(s)N(s) + k_1(s)(\tau(s)T(s) - B(s)) + k_2'(s)B(s) - k_2(s)\tau(s)N(s) \\ &= (1 + k_1(s)\tau(s))T(s) + (k_1'(s) - k_1(s))N(s) + (k_2'(s) - k_1(s))B(s).\end{aligned}$$

Since $\alpha^{*'}(s)$ is tangent to $\alpha^*(s)$, it is proportional to $\alpha^*(s) - \alpha(s) = k_1(s)N(s) + k_2(s)B(s)$. Therefore, we get

$$1 + k_1(s)\tau(s) = 0 \implies k_1(s) = -\frac{1}{\tau(s)}.$$

On the other hand, $\langle T^*(s), T(s) \rangle = 0$. Consequently,

$$\begin{aligned}0 &= \langle \alpha^{*'}(s), T(s) \rangle \\ &= (1 + k_1(s)\tau(s))\langle T(s), T(s) \rangle + (k_1'(s) - k_2(s))\langle N(s), T(s) \rangle \\ &\quad + (k_2'(s) - k_1(s))\langle B(s), T(s) \rangle \\ &= k_2'(s) - k_1(s).\end{aligned}$$

This gives us



$$k'_2(s) = k_1(s) = -\frac{1}{\tau(s)}.$$

Integrating the last equation above, we have

$$k_2(s) = \int -\frac{1}{\tau(s)} d(s) = -(\psi(s) + c)$$

for some constant c . Substituting $k_1(s)$ and $k_2(s)$ into equation (6) gives us equation (10) as desired.

Theorem 10. Let $\alpha^*(s)$ be the evolute of a null curve $\alpha(s)$ parametrized by pseudo-arc length s . Then the distance between $\alpha^*(s)$ and $\alpha(s)$ is $\frac{1}{\|\tau(s)\|}$.

Proof. From equation (10), we have

$$\alpha^*(s) - \alpha(s) = -\frac{1}{\tau(s)}N(s) - (\psi(s) + c)B(s).$$

It follows that

$$\begin{aligned} \|\alpha^*(s) - \alpha(s)\|^2 &= \left\langle -\frac{1}{\tau(s)}N(s) - (\psi(s) + c)B(s), -\frac{1}{\tau(s)}N(s) - (\psi(s) + c)B(s) \right\rangle \\ &= \frac{1}{\tau^2(s)}. \end{aligned}$$

Therefore,

$$\|\alpha^*(s) - \alpha(s)\| = \frac{1}{\|\tau(s)\|}$$

and it completes the proof.

Theorem 11. Let $\alpha^*(s)$ be the evolute of a null curve $\alpha(s)$ parametrized by pseudo-arc length s . Then $\alpha^*(s)$ is a spacelike curve.

Proof. Differentiating equation (10) yields

$$\alpha^{*'}(s) = \left(\frac{\tau'(s)}{\tau^2(s)} + \psi(s) + c \right) N(s).$$

As a consequence,

$$\begin{aligned} \langle \alpha^{*'}(s), \alpha^{*'}(s) \rangle &= \left\langle \left(\frac{\tau'(s)}{\tau^2(s)} + \psi(s) + c \right) N(s), \left(\frac{\tau'(s)}{\tau^2(s)} + \psi(s) + c \right) N(s) \right\rangle \\ &= \left(\frac{\tau'(s)}{\tau^2(s)} + \psi(s) + c \right)^2 > 0. \end{aligned}$$

Therefore $\alpha^*(s)$ is a spacelike as desired.

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