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LOCAL METRIC DIMENSION OF THE LINE GRAPH OF A GENERALIZED PETERSEN GRAPH

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Abstrak. Misalkan G adalah graf yang memiliki himpunan titik $V(G)$ dan himpunan sisi $E(G)$. Misalkan $W = \{w_1, w_2, \dots w_k\}$ adalah himpunan bagian dari $V(G)$. Representasi titik $v \in V(G)$ terhadap W, dilambangkan dengan $r(v|W)$, didefinisikan sebagai vektor-k $(d(v, w_1), d(v, w_2), \dots, d(v, w_k))$. Himpunan W disebut himpunan pembeda lokal G jika $r(u|W) \neq r(v|W)$ untuk setiap dua titik yang bertetangga $u, v \in V(G)$. Kardinalitas terkecil dari semua himpunan pembeda lokal di G disebut dimensi metrik lokal G, dilambangkan dengan $\text{lmd}(G)$. Himpunan pembeda lokal G dengan kardinalitas $\text{lmd}(G)$ disebut basis lokal dari G. Dalam artikel ini, kami menentukan dimensi metrik lokal dari graf garis dari suatu graf Petersen diperumum $P_{n,1}$.

Kata Kunci: **Dimensi Metrik Lokal, Graf Garis, Graf Petersen Diperumum.**

Abstract. Let G be a graph that has a vertex set $V(G)$ and an edge set $E(G)$. Let $W = \{w_1, w_2, \dots w_k\}$ be a subset of $V(G)$. The representation of a vertex $v \in V(G)$ with respect to W, denoted by $r(v|W)$, is defined as k-vector $(d(v, w_1), d(v, w_2), \dots, d(v, w_k))$. A set W is called a local resolving set of G if $r(u|W) \neq r(v|W)$ for every two adjacent vertices $u, v \in V(G)$. The smallest cardinality of all local resolving set in G is called the local metric dimension of G, denoted by $\text{lmd}(G)$. The local resolving set of G with cardinality $\text{lmd}(G)$ is called a local basis of G. In this paper, we determine the local metric dimension of the line graph of generalized Petersen graph $P_{n,1}$.

Keywords: **Local metric dimension, Line graph, Generalized Petersen Graph.**

A. Introduction

All the graphs discussed in this article are simple, finite, and connected. For a graph G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. Let G be a graph that has a vertex set $V(G)$ and an edge set $E(G)$. Let $u, v \in V(G)$, we write $u \sim_G v$ if u is adjacent to v in G (if there is no ambiguity about graph G, we write $u \sim v$). If u and v are not adjacent in G, we write $u \nsim_G v$.

In this article, we discuss the local metric dimension problem of a graph. This problem is another variant of the metric dimension, a problem first introduced by Harary and Melter, and separately by Slater (Slater, 1975). For the distance between two vertices u and v in G, we denoted by $d(u, v)$. Let $W = \{w_1, w_2, \dots w_k\}$ be a subset of $V(G)$. The representation of a vertex $v \in V(G)$ with respect to W, denoted by $r(v|W)$, is defined as k-vector $(d(v, w_1), d(v, w_2), \dots, d(v, w_k))$. A set W is called a local resolving set of G if $r(u|W) \neq r(v|W)$ for every two adjacent vertices $u, v \in V(G)$. The smallest cardinality of all local resolving set in G is called the local metric dimension of G, denoted by $\text{lmd}(G)$. The local resolving set of G with cardinality $\text{lmd}(G)$ is called a local basis of G.

Local metric dimension problem was first studied by Okamoto, *et al* (Okamoto et al., 2010). Okamoto has proven several statements related to the local metric dimension of a graph, one of which is the following.

Theorem 1. (Okamoto et al., 2010) *Let G be a nontrivial connected graph of order n.* $\text{lmd}(G) = n - 1$ *if and only if* $G = K_n$ *and* $\text{lmd}(G) = 1$ *if and only if G is a bipartite graph.*

Several authors have studied the local metric dimension of some graphs and have obtained some results (F & Jude Annie Cynthia, 2017)(Fitriani et al., 2022)(Solekhah & Kusmayadi, 2018).

The graph discussed in this article is the line graph of a generalized Petersen graph. A generalized Petersen graph, denoted by $P_{n,m}$ with $n \geq 3$ and $1 \leq m \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, is a graph with a vertex set $V(P_{n,m}) = \{u_1, u_2, u_3 \dots, u_n, v_1, v_2, v_3 \dots, v_n\}$ and an edge set $E(P_{n,m}) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+m}$ |with indices taken mod $n\}$. This graph was introduced by Watkins (Watkins, 1969). The line graph of graph G, denoted by $L(G)$, is a graph whose vertices are obtained from one-to-one correspondence with the edges of G. In this article, we denote a vertex of $L(G)$ by uv to mean that the vertices u and v in G are adjacent vertices in G or uv is an edge in G. Let $u, v,$ and w are in $V(G)$. Two vertices uv and vw in $L(G)$ are adjacent if only if two edges uv and vw in G are adjacent. Some result of study about the local metric dimension of the line graph can be seen in (Marsidi et al., 2016). In this study, we determine the local metric dimension of the line graph of a generalized Petersen graph $P_{n,m}$ with $m = 1$. The line graph of a generalized Petersen graph $P_{n,1}$, denoted by $L(P_{n,1})$, is a graph with vertex set $V\left(L(P_{n,1})\right) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+1}$|with indices taken mod $n\}$. The adjacent vertices in $L(P_{n,1})$ are $u_i u_{i+1} \sim u_{i+1} u_{i+2}$, $u_i u_{i+1} \sim u_{i+1} v_{i+1}$, $u_i u_{i+1} \sim u_i v_i$, $u_i v_i \sim v_{i-1} v_i$, $u_i v_i \sim v_i v_{i+1}$, and $v_i v_{i+1} \sim v_{i+1} v_{i+2}$. An example of a generalized Petersen graph and its line graph are represented in Figure 1.

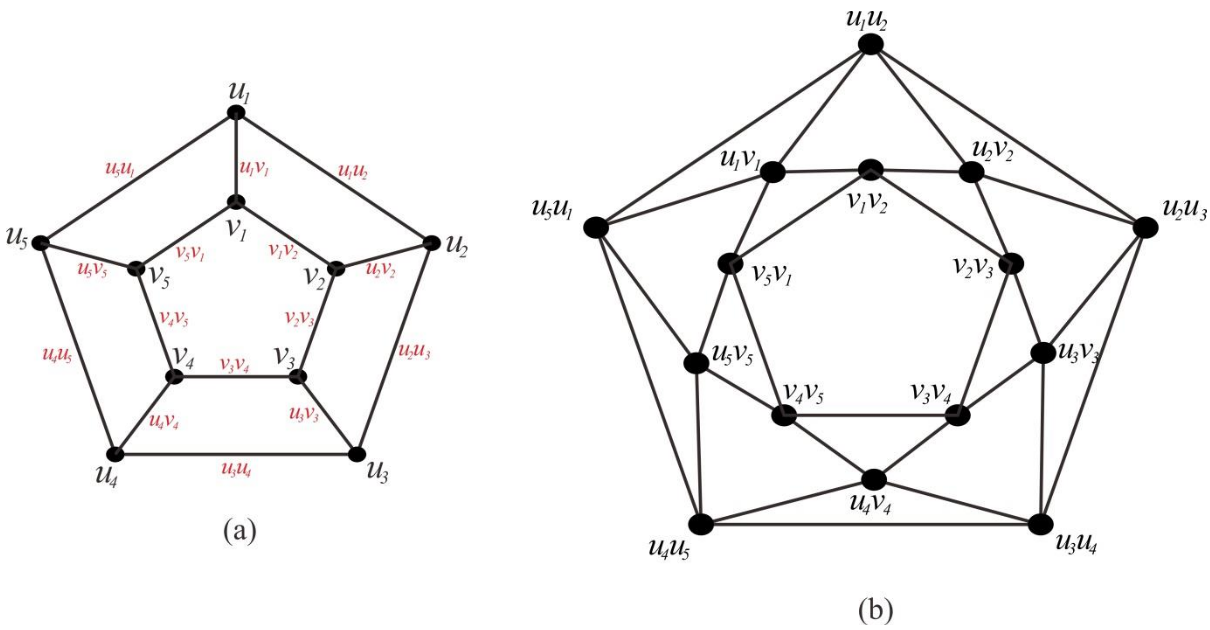

Figure 1. Generalized Petersen graph $P_{5,1}$ (a) and its line graph (b)

B. Method

This research is a literature study. Several references related to local metric dimension are collected and studied. Several definitions and results of previous studies are used to prove the results of this study. In determining the local metric dimension of $L(P_{n,1})$, a subset of $V(L(P_{n,1}))$ is proven to be a local resolving set. Furthermore, we show that that subset is a local basis of $L(P_{n,1})$. From there, we determine the local metric dimension of $L(P_{n,1})$.

C. Result and Discussion

In this section, we determine the local metric dimension of the line graph of a generalized Petersen graph $P_{n,1}$.

Theorem 2. *Let $L(P_{n,1})$ be a line graph of a generalized Petersen graph $P_{n,1}$ with $n \geq 5$. Then* $\text{lmd}(L(P_{n,1})) = 2$.

Proof.

Note that $L(P_{n,1})$ contains a complete subgraph K_3 of order 3, so $L(P_{n,1})$ is not a bipartite graph. By theorem 1, $\text{lmd}(L(P_{n,1})) \geq 2$. Next, we consider two cases about n.

Case 1: n is even

Let W be a subset of $V(L(P_{n,1}))$ and $W = \{u_1u_2, u_ku_{k+1}\}$ with $k = \frac{n+2}{2}$. For every two adjacent vertices in $L(P_{n,1})$, their representations with respect to W are shown as follows:

1. u_iu_{i+1} and $u_{i+1}u_{i+2}$
$$r(u_iu_{i+1}|W) = \begin{cases} (i-1, k-i), & \text{if } 2 \leq i \leq k-2 \\ (2k-(i+1), i-k), & \text{if } k+1 \leq i \leq 2k-3 \end{cases}$$
and
$$r(u_{i+1}u_{i+2}|W) = \begin{cases} (i, k-(i+1)), & \text{if } 2 \leq i \leq k-2 \\ (2k-(i+2), (i+1)-k), & \text{if } k+1 \leq i \leq 2k-3. \end{cases}$$
2. u_iu_{i+1} and $u_{i+1}v_{i+1}$
$$r(u_iu_{i+1}|W) = \begin{cases} (i-1, k-i), & \text{if } 2 \leq i \leq k-1 \\ (2k-(i+1), i-k), & \text{if } k+1 \leq i \leq 2k-2 \end{cases}$$
and
$$r(u_{i+1}v_{i+1}|W) = \begin{cases} (i, k-i), & \text{if } 2 \leq i \leq k-1 \\ (2k-(i+1), i-k+1), & \text{if } k+1 \leq i \leq 2k-2. \end{cases}$$
3. u_iu_{i+1} and u_iv_i
$$r(u_iu_{i+1}|W) = \begin{cases} (i-1, k-i), & \text{if } 2 \leq i \leq k-1 \\ (2k-(i+1), i-k), & \text{if } k+1 \leq i \leq 2k-2 \end{cases}$$
and
$$r(u_iv_i|W) = \begin{cases} (i-1, k-i+1), & \text{if } 2 \leq i \leq k-1 \\ (2k-i, i-k), & \text{if } k+1 \leq i \leq 2k-2. \end{cases}$$
4. u_iv_i and $v_{i-1}v_i$
$$r(u_iv_i|W) = \begin{cases} (i-1, k-i+1), & \text{if } 3 \leq i \leq k \\ (2k-i, i-k), & \text{if } k+2 \leq i \leq 2k-2 \end{cases}$$
and
$$r(v_{i-1}v_i|W) = \begin{cases} (i-1, k-i+2), & \text{if } 3 \leq i \leq k \\ (2k-i+1, i-k), & \text{if } k+2 \leq i \leq 2k-2. \end{cases}$$

For $u_1v_1 \sim v_{2k-2}v_1$, $u_2v_2 \sim v_1v_2$, and $u_{k+1}v_{k+1} \sim v_kv_{k+1}$, we obtain that $r(u_1v_1|W) = (1, k-1) \neq (2, k-1) = r(u_{2k-2}v_1|W)$, $r(u_2v_2|W) = (1, k-1) \neq (2, k) = r(v_1v_2|W)$, and $r(u_{k+1}v_{k+1}|W) = (k-1, 1) \neq (k, 2) = r(v_kv_{k+1}|W)$.

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5. u_iv_i and v_iv_{i+1}

$$r(u_iv_i|W) = \begin{cases} (i-1, k-i+1), & \text{if } 2 \leq i \leq k-1 \\ (2k-i, i-k), & \text{if } k+1 \leq i \leq 2k-2 \end{cases}$$

and

$$r(u_iv_{i+1}|W) = \begin{cases} (i, k-i+1), & \text{if } 2 \leq i \leq k-1 \\ (2k-i, i-k+1), & \text{if } k+1 \leq i \leq 2k-2. \end{cases}$$

For $u_1v_1 \sim v_1v_2$ and $u_kv_k \sim v_kv_{k+1}$, we obtain that $r(u_1v_1|W) = (1, k-1) \neq (2, k) = r(v_1v_2|W)$ and $r(u_kv_k|W) = (k-1, 1) \neq (k, 2) = r(v_kv_{k+1}|W)$.

6. v_iv_{i+1} and $v_{i+1}v_{i+2}$

$$r(v_iv_{i+1}|W) = \begin{cases} (i, k-i+1), & \text{if } 2 \leq i \leq k-2 \\ (2k-i, i-k+1), & \text{if } k+1 \leq i \leq 2k-3 \end{cases}$$

and

$$r(v_{i+1}v_{i+2}|W) = \begin{cases} (i+1, k-i), & \text{if } 2 \leq i \leq k-2 \\ (2k-i-1, i-k+2), & \text{if } k+1 \leq i \leq 2k-3. \end{cases}$$

For $v_{k-1}v_k \sim v_kv_{k+1}$ and $v_{2k-2}v_1 \sim v_1v_2$, we obtain that $r(v_{k-1}v_k|W) = (k-1, 2) \neq (k, 2) = r(v_kv_{k+1}|W)$ and $r(v_{2k-2}v_1|W) = (2, k-1) \neq (2, k) = r(v_1v_2|W)$.

Note that every two adjacent vertices in $L(P_{n,1})$ have different representations with respect to W. Therefore, $\text{lmd}\left(L(P_{n,1})\right) \leq 2$ if n is even.

Case 2: n is odd

Let W be a subset of $V(L(P_{n,1}))$ and $W = \{u_1v_1, u_kv_k\}$ with $k = \frac{n+1}{2}$. For every two adjacent vertices in $L(P_{n,1})$, their representations with respect to W are shown as follows:

1. u_iu_{i+1} and $u_{i+1}u_{i+2}$

$$r(u_iu_{i+1}|W) = \begin{cases} (i, k-i), & \text{if } 1 \leq i \leq k-2 \\ (2k-i, i-k+1), & \text{if } k \leq i \leq 2k-2 \end{cases}$$

and

$$r(u_{i+1}u_{i+2}|W) = \begin{cases} (i+1, k-i-1), & \text{if } 1 \leq i \leq k-2 \\ (2k-i-1, i-k+2), & \text{if } k \leq i \leq 2k-2. \end{cases}$$

For $u_{k-1}u_k \sim u_ku_{k+1}$ and $u_{2k-1}u_1 \sim u_1u_2$, we obtain that $r(u_{k-1}u_k|W) = (k-1, 1) \neq (k, 1) = r(u_ku_{k+1}|W)$ and $r(u_{2k-1}u_1|W) = (1, k) \neq (1, k-1) = r(u_1u_2|W)$.

2. u_iu_{i+1} and $u_{i+1}v_{i+1}$

$$r(u_iu_{i+1}|W) = \begin{cases} (i, k-i), & \text{if } 1 \leq i \leq k-2 \\ (2k-i, i-k+1), & \text{if } k \leq i \leq 2k-2 \end{cases}$$

and

$$r(u_{i+1}v_{i+1}|W) = \begin{cases} (i+1, k-i), & \text{if } 1 \leq i \leq k-2 \\ (2k-i, i-k+2), & \text{if } k \leq i \leq 2k-2. \end{cases}$$

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3. u_iu_{i+1} and u_iv_i

$$r(u_iu_{i+1}|W) = \begin{cases} (i, k-i), & \text{if } 2 \le i \le k-1 \\ (2k-i, i-k+1), & \text{if } k+1 \le i \le 2k-1 \end{cases}$$

and

$$r(u_iv_i|W) = \begin{cases} (i, k-i+1), & \text{if } 2 \le i \le k-1 \\ (2k-i+1, i-k+1), & \text{if } k+1 \le i \le 2k-1. \end{cases}$$

4. u_iv_i and $v_{i-1}v_i$

$$r(u_iv_i|W) = \begin{cases} (i, k-i+1), & \text{if } 2 \le i \le k-1 \\ (2k-i+1, i-k+1), & \text{if } k+1 \le i \le 2k-1 \end{cases}$$

and

$$r(v_{i-1}v_i|W) = \begin{cases} (i-1, k-i+1), & \text{if } 2 \le i \le k-1 \\ (2k-i+1, i-k), & \text{if } k+1 \le i \le 2k-1. \end{cases}$$

5. u_iv_i and v_iv_{i+1}

$$r(u_iv_i|W) = \begin{cases} (i, k-i+1), & \text{if } 2 \le i \le k-1 \\ (2k-i+1, i-k+1), & \text{if } k+1 \le i \le 2k-1 \end{cases}$$

and

$$r(u_iv_{i+1}|W) = \begin{cases} (i, k-i), & \text{if } 2 \le i \le k-1 \\ (2k-i, i-k+1), & \text{if } k+1 \le i \le 2k-1. \end{cases}$$

6. v_iv_{i+1} and $v_{i+1}v_{i+2}$

$$r(v_iv_{i+1}|W) = \begin{cases} (i, k-i), & \text{if } 1 \le i \le k-2 \\ (2k-i, i-k+1), & \text{if } k \le i \le 2k-2 \end{cases}$$

and

$$r(v_{i+1}v_{i+2}|W) = \begin{cases} (i+1, k-i-1), & \text{if } 1 \le i \le k-2 \\ (2k-i-1, i-k+2), & \text{if } k \le i \le 2k-2. \end{cases}$$

For $v_{k-1}v_k \sim v_kv_{k+1}$ and $v_{2k-1}v_1 \sim v_1v_2$, we obtain that $r(v_{k-1}v_k|W) = (k-1,1) \neq (k,1) = r(v_kv_{k+1}|W)$ and $r(v_{2k-1}v_1|W) = (1,k) \neq (1,k-1) = r(v_1v_2|W)$.

Similar to case 1, we obtain that every two adjacent vertices in $L(P_{n,1})$ have different representations with respect to W when n is odd. So, $\text{lmd}(L(P_{n,1})) \le 2$ if n is odd.

Since $\text{lmd}(L(P_{n,1})) \le 2$, we conclude that $\text{lmd}(L(P_{n,1})) = 2$. ■

In previous studies, the local metric dimension of the line graphs of several special graphs, such as path, cycle, generalized star, and wheel, has been determined. Those study do not cover the line graph of the Petersen graph. In this study, we obtain the local metric dimension of the line graph of $P_{n,1}$.

D. Conclusion and Suggestion

The local metric dimension of the line graph of generalized Petersen graph $P_{n,1}$ is 2. This result, especially the determination of the local basis of $L(P_{n,1})$, can be used as consideration in determining the local metric dimension of the line graph of certain operation for a generalized Petersen graph $P_{n,1}$ as defined in (Asmiati et al., 2020). This requires further study.

REFERENCES

Asmiati, A., Aldino, A. A., Notiragayu, N., Zakaria, L., & Anshori, M. (2020). Dimensi Metrik Hasil Operasi Tertentu pada Graf Petersen Diperumum. *Limits: Journal of Mathematics and Its Applications*, *16*(2), 87. https://doi.org/10.12962/limits.v16i2.5594

F, F. V, & Jude Annie Cynthia, V. (2017). Local Metric Dimension of Families of Certain Graphs. *Journal of Combinatorics, Information & System Sciences*, *42*(1), 179–186.

Fitriani, D., Rarasati, A., Saputro, S. W., & Baskoro, E. T. (2022). The local metric dimension of split and unicyclic graphs. *Indonesian Journal of Combinatorics*, *6*(1), 50. https://doi.org/10.19184/ijc.2022.6.1.3

Marsidi, Dafik, Agustin, I. H., & Alfarisi, R. (2016). On The Local Metric Dimension of Line Graph of Special Graph. *CAUCHY – JURNAL MATEMATIKA MURNI DAN APLIKASI*, *4*(3), 125–130.

Okamoto, F., Phinezy, B., & Zhang, P. (2010). The local metric dimension of a graph. *Mathematica Bohemica*, *135*(3), 239–255. https://doi.org/10.21136/mb.2010.140702

Slater, P. J. (1975). Leaves of trees. *Proc. 6 Th Southeastern Conf. on Combinatorics, Graph Theory, and Computing, Congressus Numerantium*, *14*, 549–559.

Solekhah, R. A., & Kusmayadi, T. A. (2018). On the local metric dimension of t-fold wheel, Pn o Km, and generalized fan. *Indonesian Journal of Combinatorics*, *2*(2), 88. https://doi.org/10.19184/ijc.2018.2.2.4

Watkins, M. E. (1969). A theorem on tait colorings with an application to the generalized Petersen graphs. *Journal of Combinatorial Theory*, *6*(2), 152–164. https://doi.org/10.1016/S0021-9800(69)80116-X