Jurnal Penelitian Matematika dan Pendidikan Matematika
ISSN 26158132 (cetak)
ISSN 26157667 (online)

MODULAR IRREGULAR LABELING ON FIRECRACKERS GRAPHS

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Abstract Let G= (V, E) be a graph order n and an edge labeling ψ: E→{1,2,…,k}. Edge k labeling ψ is to be modular irregular -k labeling if exist a bijective map σ: V→Zn with σ(x)= ∑yϵv ψ(xy)(mod n). The modular irregularity strength of G (ms(G))is a minimum positive integer k such that G have a modular irregular labeling. If the modular irregularity strength is none, then it is defined ms(G) = ∞. Investigating the firecrackers graph (Fn,2), we find irregularity strength of firecrackers graph s(Fn,2), which is also the lower bound for modular irregularity strength, and then we construct a modular irregular labeling and find modular irregularity strength of firecrackers graph ms(Fn,2). The result shows its irregularity strength and modular irregularity strength are equal.

Keywords: **Firecracker graph; Irregular labeling; Modular irregular labeling; Modular irregularity strength**

A. Introduction

Let G= (V, E) be a graph order n. A graph is called irregular if no two of its vertices have the same degree. By adding multiple edges to G, each vertex can have distinct degree. It means that multigraph can have that property. Replacing multiple edges joining each pair of vertices by its number, (Chartrand et al, 1988) introduced the well-known labeling of G, called the irregular assignment, it is an edge k-labeling of the edge-set $\varphi: E(G) \rightarrow \{1, 2, \cdots, k\}$ such that the vertex-weights are all distinct, where the weight of a vertex x is the sum of all labels of edges incident to x, wrote $\omega t_{\varphi}(x) = \sum_{y \in V} \varphi(xy)$. The minimum k for which G has a vertex irregular edge k-labeling is called the irregularity strength of G, denoted by $s(G)$. If a G admit no -k labeling, then $s(G) = \infty$. Chartrand et al. in (Chartrand et al, 1988) give a lower bound on the graph as follows

$$s(G) \geq \max_{1 \leq i \leq \Delta} \left\{\frac{n_i + i - 1}{i}\right\}, \quad (1)$$

where n_i is the number of vertices of degree i and Δ is the maximum degree of G. In 2020, (Baca et al, 2020) introduced new irregular labeling which is modified. it is a modular irregular labeling. The labeling edge $\psi : E(G) \rightarrow \{1,2,\dots,k\}$ is a modular irregular k-labeling of G if there exists a bijective weight function $\sigma : V(G) \rightarrow Z_n$ with

$$\sigma(x) = \sum_{y \in V} \psi(xy) \ (mod\ n), \quad (2)$$

where $x, y \in V(G)$, Z_n is the set of integers modulo n and $\sigma(x)$ is the sum of the labels of all the vertices adjacent to the vertex x. The minimum k of a graph G which is a k −labeling modular irregular is called the value of the modular irregularity of G denoted by $ms(G)$. If there is no modular labeling for the graph G, then is defined $ms(G) = \infty$ (Muthugurupackiam et al., 2020).

Now, the lower bound of the modular irregularity strength of a graph G no component of order ≤ 2 is given in (Muthugurupackiam et al., 2020) as follow:

$$s(G) \leq ms(G). \quad (3)$$

https://doi.org/10.30605/proximal.v5i2.2188

Subsequently, in (Muthugurupackiam et al., 2020) explain that infinity condition for the modular irregularity strength of a graph by

Theorem 1. If G is a graph of order n, $n \equiv 2(\text{mod } 4)$ then G has no modular irregular k −labeling i.e., $ms(G) = \infty$.

Muthugurupackiam et al., (2020) elaborate the value of modular irregularity strength of path graph, star graph, triangular graph, cycle graph, and gear graph. Then, (Baca et al, 2021) explain modular irregularity strength of the fan graph. (Muthugurupackiam et al., 2020) explain modular irregularity strength of tadpole graphs and double-cycle graphs. In (Baca, , Imran, & Fenovcikova, 2021). determine the value of the irregularity of the wheel graph. Latest, in 2021, (Sugeng et al, 2021) determines modular irregularity strength double star graph and a friendly graph. At the last, (Tilukay, 2021) explain the modular irregularity strength of triangular book graph.

In this paper, we determine the value of the irregularity strength and modular irregularity strength of the firecracker graph $F_{n,2}$.

B. Result and Discussion

In this section, we will describe the results of modular irregular labeling on firecracker graph $(F_{n,2})$. In addition, it will also show the lower bound of the strength irregularity in the firecracker graph $(F_{n,2})$ according to Theorem 1.

Description of firecracker graph $(F_{n,2})$

In this section, we describe the firecracker graph $(F_{n,2})$.

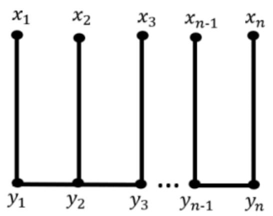

Figure 1. Firecracker Graph $F_{n,2}$

According to Figure 1 Firecracker Graph $F_{n,2}$, then we define the set of vertices and edges of firecracker graph $F_{n,2}$, for $n \geq 2$ as follows

$$V(F_{n,2}) = \{x_1, \dots, x_n, y_1, \dots, y_n\} = \{x_i, y_i | i = 1, \dots, n\}, \quad (4)$$

$$E(F_{n,2}) = \{x_1y_1, \dots, x_ny_n, y_1y_2, \dots, y_{n-1}y_n\} = \{x_iy_i, y_iy_{i+1} | i = 1, \dots, n-1\} \cup \{x_ny_n\}. \quad (5)$$

So, firecracker graph $F_{n,2}$ has $2n$ vertices and $2n - 1$ edges.

Irregularity Strength of firecracker graph $(F_{n,2})$, n even

In this section, we discuss the irregularity strength for firecracker graph $(F_{n,2})$. We construct an edge labeling and show that this labeling meets the required properties.

Proposition 1. For $n = 2$, irregularity strength for firecracker graph $s(F_{n,2}) = 2$

Proof:

To determine the weight vertex irregularity of firecracker graph $(F_{n,2})$, Figure 2 shows k −labeling of the edge irregularity of firecracker graph $(F_{n,2})$ as follows:

https://doi.org/10.30605/proximal.v5i2.2188

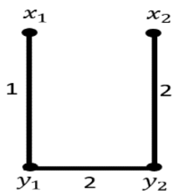

Figure 2. Firecracker Graph ($F_{2,2}$)

In Figure 2 shows that it is an irregular - 2 labeling and Table 1 shows that the vertex weights are all different.

Table 1 Irregular - 2 labeling firecracker graph $F_{2,2}$

Vertex	x_1	x_2	y_1	y_2
x_1	0	0	1	0
x_2	0	0	0	2
y_1	1	0	0	2
y_2	0	2	2	0
Weights	1	2	3	4

According to Table 1 it can be seen that the edge labeling has function $\varphi: E \to \{1,2\}$ so that all the weights at each vertex are different and at the same time show the value $k = 2$. Thus, it can be concluded that $s(F_{2,2}) = 2$.

Proposition 2. For $n > 2$, irregularity strength for firecracker graph $s(F_{n,2}) = n$

Proof:

To prove preposition 2, it is divided into 2 cases described as follows:

Case 1: For $n = 4$

We define the edges labeling φ as follows:

For edge $x_i y_i$,

$$\varphi(x_i y_i) = i \,, \; 1 \leq i \leq 4 \tag{6}$$

For edge $y_i y_{i+1}$

$$\varphi(y_i y_{i+1}) = \begin{cases} n, & i = 1 \\ 2, & 1 < i < n \end{cases} \tag{7}$$

Based on labeling φ, we find the weights of vertices as follows:

For vertex x_i

$$\omega t_\varphi(x_i) = \varphi(x_i y_i)$$
$$= i, \quad 1 \leq i \leq 4$$

(8)

For vertex y_i, $1 \leq i \leq 4$ (9)

a. $\omega t_\varphi(y_1) = \varphi(x_1 y_1) + \varphi(y_1 y_2)$
$= 1 + 4$
$= 5$

b. $\omega t_\varphi(y_2) = \varphi(x_2 y_2) + \varphi(y_1 y_2) + \varphi(y_2 y_3)$
$= 2 + 4 + 2$
$= 8$

c. $\omega t_\varphi(y_3) = \varphi(x_3 y_3) + \varphi(y_2 y_3) + \varphi(y_3 y_4)$

$$= 3 + 2 + 2$$
$$= 7$$

d. $\omega t_\varphi(y_4) = \varphi(x_4y_4) + \varphi(y_3y_4)$
$$= 4 + 2$$
$$= 6$$

Base on (8) and (9) it is found that all the weights at each vertex are different from edges labeling $\varphi: E \to \{1,2,3,4\}$. The value $k = 4$ is largest label and at the same time the minimum positive integer k such that firecracker graph $(F_{4,2})$ has an irregular -4 labeling. Thus, irregularity strength for firecracker graph $s(F_{4,2}) = 4$.

Case 2: For $n > 2, n \neq 4$

We define the edges labeling φ as follows:

For edge x_iy_i,

$$\varphi(x_iy_i) = i, \quad 1 \le i \le n \tag{10}$$

For edge y_iy_{i+1}, i odd,

$$\varphi(y_iy_{i+1}) = \begin{cases} n, & i = 1 \\ n-2, & 1 < i < n-1 \\ n-1, & i = n-1 \end{cases} \tag{11}$$

For edge y_iy_{i+1}, i even,

$$\varphi(y_iy_{i+1}) = \begin{cases} 1, & i = 2 \\ 2, & 2 < i < n \end{cases} \tag{12}$$

Based on labeling φ, we find the weights of vertices as follows:

For vertex x_i

$$\omega t_\varphi(x_i) = \varphi(x_iy_i),$$
$$= i \quad 1 \le i \le n \tag{13}$$

For vertex y_i, i odd

a. $\omega t_\varphi(y_1) = \varphi(x_1y_1) + \varphi(y_1y_2)$
$$= 1 + n,$$

b. $\omega t_\varphi(y_3) = \varphi(x_3y_3) + \varphi(y_2y_3) + \varphi(y_3y_4)$
$$= 3 + 1 + n - 2$$
$$= n + 2,$$

c. $\omega t_\varphi(y_i) = \varphi(x_iy_i) + \varphi(y_iy_{i+1}) + \varphi(y_{i-1}y_i)$
$$= i + n - 2 + 2$$
$$= i + n, \quad 3 < i < n - 1$$

d. $\omega t_\varphi(y_{n-1}) = \varphi(x_{n-1}y_{n-1}) + \varphi(y_{n-1}y_n) + \varphi(y_{n-2}y_{n-1})$
$$= i + n - 1 + 2$$
$$= n - 1 + n + 1$$
$$= 2n, \quad i = n - 1$$

From the descriptions (a), (b), (c) and (d) it can be concluded $\omega t_\varphi(y_i)$, i odd as follows:

$$\omega t_\varphi(y_i) = \begin{cases} 1 + n, & i = 1 \\ n + 2, & i = 3 \\ i + n, & 3 < i < n - 1 \\ 2n, & i = n - 1 \end{cases} \tag{14}$$

For vertex y_i, i even

e. $\omega t_\varphi(y_2) = \varphi(x_2y_2) + \varphi(y_2y_3) + \varphi(y_1y_2)$

$= 2 + 1 + n$

$= n + 3, \quad i = 2$

f. $\omega t_{\varphi}(y_i) = \varphi(x_i y_i) + \varphi(y_i y_{i+1}) + \varphi(y_{i-1} y_i)$

$= i + 2 + n - 2$

$= i + n, \quad 2 < i < n$

g. $\omega t_{\varphi}(y_n) = \varphi(x_n y_n) + \varphi(y_{n-1} y_n)$

$= i + n - 1$

$= 2n - 1, \quad i = n$

From the descriptions (e), (f) and (g) it can be concluded $\omega t_{\varphi}(y_i)$, i even as follows:

$$\omega t_{\varphi}(y_i) = \begin{cases} n+3, & i = 2 \\ i + n, & 2 < i < n-2 \\ 2n-1, & i = n \end{cases} \quad (15)$$

Based on (13), (14) and (15), it is found that all the weights at each vertex are different as follows:

$$\omega t_{\varphi}(x_i) = i \quad \text{for } 1 \leq i \leq n \quad (17)$$

$$\omega t_{\varphi}(y_i) = \begin{cases} n+1, & i = 1 \\ n+3 & i = 2 \\ n+2 & i = 3 \\ i + n, & 2 < i < n-1 \\ 2n, & i = n-1 \\ 2n-1, & i = n \end{cases} \quad (18)$$

Based on (17) and (18) it is found that all the weights at each vertex are different from edges labeling $\varphi: E \rightarrow \{1,2,\dots,(k = n)\}$. The value $k = n$ is largest label and at the same time the minimum positive integer k such that firecracker graph $(F_{n,2})$ has an irregular $-n$ labeling. Thus, irregularity strength for firecracker graph $s(F_{n,2}) = n,\ n \geq 2\ even$.

■

Modular Irregularity Strength of firecracker graph $(F_{n,2})$, n even

In this section, we discuss the modular irregularity strength for firecracker graph $(F_{n,2})$. We construct an edge labeling and show exact value of modular irregularity strength for firecracker graph $(F_{n,2})$. The results in this section will be used as conclusions.

Proposition 3. Let $F_{n,2}$ be a firecracker graph, $n \geq 2$, then

$$ms(F_{n,2}) \geq n \quad (19)$$

Proof:

A firecracker graph $F_{n,2}$ has n vertices with degree 1, $n - 2$ vertices with degree 3, and 2 vertices with degree 2. Based on (1), we have

$$s(F_{n,2}) \geq \left\{\frac{(n)+1-1}{1}, \frac{(n-2)+3-1}{3}, \frac{(2)+2-1}{2},\ 1 \leq i \leq n\right\} \quad (20)$$

$$s(F_{n,2}) \geq \{n\} \text{ (Since } s(F_{n,2}) \text{ is an integer)},$$

$$s(F_{n,2}) \geq \left\{\frac{n}{3}\right\},$$

$$s(F_{n,2}) \geq \left\{\frac{3}{2}\right\}.$$

Based on (3), we have

$$ms(F_{n,2}) \geq s(F_{n,2}) \geq n$$

$$ms(F_{n,2}) \geq n.$$

Theorem 2. For $n \geq 2$, modular irregularity strength for firecracker graph $(F_{n,2})$

$$ms(F_{n,2}) = n \quad (21)$$

Proof:

To prove theorem 2, it is divided into 2 cases described as follows:

Case 1: For $n = 4$

According to (4) and (5), then firecracker graph $F_{4,2}$ has $|V(F_{4,2})| = 2(4) = 8\ vertices$ and $|E(F_{4,2})| = 2(4) - 1 = 7\ edges$. Notice Figure 3. Firecracker graph $(F_{4,2})$ as follows:

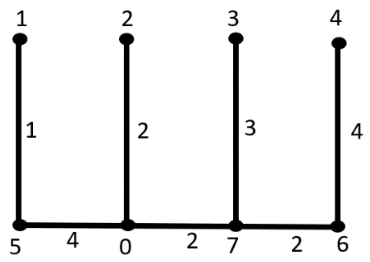

Figure 3 Firecracker Graph $(F_{4,2})$

We define the edges labeling ψ as follows:

For edge $x_i y_i$

$$\psi(x_i y_i) = i, \quad 1 \leq i \leq 4 \quad (22)$$

For edge $y_i y_{i+1}$

$$\psi(y_i y_{i+1}) = \begin{cases} 1, & i = 1 \\ 2, & 2 \leq i \leq n-1 \end{cases} \quad (23)$$

Based on labeling ψ, we find the weights of vertices $\sigma(x) = \sum_{y \in V} \psi(xy)\ (mod\ 2n)$ as follows:

For vertex x_i

$$\begin{aligned} \sigma(x_i) &= \psi(x_i y_i), \\ &= i\ (mod\ 2n), \quad 1 \leq i \leq 4 \end{aligned} \quad (24)$$

For vertex y_i, i odd (25)

$$\begin{aligned} \sigma(y_i) &= \psi(x_i y_i) + \psi(y_i y_{i+1}) \\ &= i + n \\ &= n + 1 (mod\ 2n), \quad i = 1 \\ \sigma(y_i) &= \psi(x_i y_i) + \psi(y_i y_{i+1}) + \psi(y_{i-1} y_i) \\ &= i + 2 + 2 \\ &= 7\ (mod\ 2n), \quad i = 3 \end{aligned}$$

For vertex y_i, i even (26)

$$\begin{aligned}\sigma(y_i) &= \psi(x_iy_i) + \psi(y_iy_{i+1}) + \psi(y_{i-1}y_i)\\ &= i + 2 + n\\ &= n + 4\ (mod\ 2n), \quad i = 2\\ \sigma(y_i) &= \psi(x_iy_i) + \psi(y_{i-1}y_i)\\ &= i + 2\\ &= n + 2\ (mod\ 2n), \quad i = n\end{aligned}$$

Based on (24), (25) and (26), we get all vertices are different as follows:

$\sigma(x_1) \to 1$	$\sigma(y_1) \to 5$
$\sigma(x_2) \to 2$	$\sigma(y_2) \to 0$
$\sigma(x_3) \to 3$	$\sigma(y_3) \to 7$
$\sigma(x_4) \to 4$	$\sigma(y_4) \to 6$

The set of vertices $v(F_{4,2}) = \{0,1,2,3,4,5,6,7\}$. According to the result, there exists a bijective weight function $\sigma: V(F_{4,2}) \to Z_8$. Thus, we proved and conclude that modular irregularity strength of firecracker graph $ms(F_{4,2}) = 4$.

Case 2: For $n \geq 2, n \neq 4$

We define the edges labeling ψ as follows:

For edge x_iy_i

$$\psi(x_iy_i) = i, \quad 1 \leq i \leq n \tag{27}$$

For edge y_iy_{i+1}, i odd

$$\psi(y_iy_{i+1}) = \begin{cases} n, & i = 1\\ n-2, & 1 < i < n-1\\ n-1, & i = n-1\end{cases} \tag{28}$$

For edge y_iy_{i+1}, i even

$$\psi(y_iy_{i+1}) = \begin{cases} 1, & i = 1\\ 2, & i \neq n\end{cases} \tag{29}$$

Based on labeling ψ, we find the weights of vertices $\sigma(x) = \sum_{y \in V} \psi(xy)\ (mod\ 2n)$ as follows:

For vertex x_i

$$\begin{aligned}\sigma(x_i) &= \psi(x_iy_i)\\ &= i\ (mod\ 2n), \quad 1 \leq i \leq n\end{aligned} \tag{30}$$

For vertex y_i, i odd

a. $$\begin{aligned}\sigma(y_1) &= \psi(x_1y_1) + \psi(y_1y_2)\\ &= n + 1\ (mod\ 2n), \quad i = 1\end{aligned}$$

b. $$\begin{aligned}\sigma(y_3) &= \psi(x_3y_3) + \psi(y_2y_3) + \psi(y_3y_4)\\ &= 3 + 1 + n - 2\\ &= n + 2\ (mod\ 2n), \quad i = 1\end{aligned}$$

c. $$\begin{aligned}\sigma(y_i) &= \psi(x_iy_i) + \psi(y_iy_{i+1}) + \psi(y_{i-1}y_i)\\ &= i + n - 2 + 2\\ &= i + n\ (mod\ 2n), \quad 1 < i < n-1\end{aligned}$$

d. $\sigma(y_i) \quad = \psi(x_i y_i) + \psi(y_i y_{i+1}) + \psi(y_{i-1} y_i)$
$= i + n - 1 + 2$
$= 2n \ (mod\ 2n), \quad i = n - 1$

From the descriptions (a), (b), (c) and (d) it can be concluded $\sigma(y_i)$, i odd as follows:

$$\sigma(y_i) = \begin{cases} n + 1 \ (mod\ 2n), & i = 1 \\ n + 2 \ (mod\ 2n) & i = 3 \\ i + n \ (mod\ 2n), & 1 < i < n - 1 \\ 2n \ (mod\ 2n), & i = n - 1 \end{cases} \tag{31}$$

For vertex y_i, i even

e. $\sigma(y_i) \quad = \psi(x_i y_i) + \psi(y_i y_{i+1}) + \psi(y_{i-1} y_i)$
$= i + 1 + n$
$= n + 3 \quad (mod\ 2n), \quad i = 2$

f. $\sigma(y_i) \quad = \psi(x_i y_i) + \psi(y_i y_{i+1}) + \psi(y_{i-1} y_i)$
$= i + 2 + n - 2$
$= i + n \ (mod\ 2n), \quad 2 < i < n$

g. $\sigma(y_i) \quad = \psi(x_i y_i) + \psi(y_{i-1} y_i)$
$= i + n - 1$
$= 2n - 1 \ (mod\ 2n), \ i = n$

From the descriptions (e), (f) and (g) it can be concluded $\sigma(y_i)$, i odd as follows:

$$\sigma(y_i) = \begin{cases} n + 3 \ (mod\ 2n), & i = 2 \\ i + n \ (mod\ 2n), & 2 < i < n - 2 \\ 2n - 1 \ (mod\ 2n), & i = n \end{cases} \tag{32}$$

Based on (30), (31) and (32), it is found that all the weights at each vertex are different as follows:

$$\sigma(x_i) = i \ (mod\ 2n), \quad 1 \le i \le n \tag{33}$$

$$\sigma(y_i) = \begin{cases} n + 1 \ (mod\ 2n), & i = 1 \\ n + 3 \ (mod\ 2n), & i = 2 \\ n + 2 \ (mod\ 2n), & i = 3 \\ i + n \ (mod\ 2n), & 2 < i < n - 1 \\ 2n \ (mod\ 2n), & i = n - 1 \\ 2n - 1 (mod\ 2n), & i = n \end{cases} \tag{34}$$

The irregular edges labeling $\psi: E \rightarrow \{1,2, \dots, n\}$ causes the set of vertices $\sigma = \{1,2, \dots, 2n(mod\ 2n)\}$, So there exist a modular irregular $-n$ labeling. According to the result, there exists a bijective weight function $\sigma: V(F_{n,2}) \rightarrow Z_{2n}$. Thus, we proved and conclude that modular irregularity strength of firecracker graph $ms(F_{n,2}) = n$.

Modular irregularity strength for firecracker graph $(F_{n,2})$, n ≥ 3 odd

Theorem 3. For $n \ge 3$ odd, modular irregularity strength for firecracker graph $(F_{n,2})$

$$ms(F_{n,2}) = \infty \tag{35}$$

Proof:

Firecracker graph $(F_{n,2})$ show that it has $|V(F_{n,2})| = 2n$ vertices and has $|E(F_{n,2})| = 2n-1$ edges. So that for each odd n, we obtain $|V(F_{n,2})| = 2n \equiv 2 \ (mod\ 4)$. Based on Theorem 1, if G is a graph of order n, $n \equiv 2 \pmod 4$ then G has no modular irregular k −labeling so that $ms(G) = \infty$. Thus, we conclude that for every odd $n \geq 3$, we obtain $ms(G) = \infty$. ■

C. Conclusions And Suggestions

In this paper, we determine the exact value of the irregularity strength and modular irregularity strength of firecracker graph $(F_{n,2})$, $n \geq 2$. We conclude $s(F_{n,2})$= $ms(F_{n,2})$ as follows:

$$\boldsymbol{ms(F_{n,2}) = s(F_{n,2}) = \begin{cases} n, & n \not\equiv 1 \ mod\ 2 \\ \infty, & n \equiv 2 \ mod\ 2 \end{cases}}$$

Problem 1. *Find exact value of the irregularity strength and modular irregularity strength of firecracker graph $(F_{n,m})$ with varying n and m.*

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